Non-perturbative insights from topological wormholes

Jeffrey Morais[♘,♙]

[♘] *Department of Physics, McGill University, Montréal, Québec, Canada*
[♙] *Department of Physics, University of Alberta, Edmonton, Alberta, Canada*

E-mail: jeffrey.morais@mail.mcgill.ca

Abstract: Having a global definition of a theory is required for non-local structures such as wormholes which prevent the system's Hilbert space from factorizing over the bulk spacetime. While these wormhole structures usually occur in Lorentzian spacetime manifolds, they can emerge in quantized symplectic spaces as *topological wormholes* which connect different particle or entanglement orbits. Topological wormholes affect the dynamics of the system non-perturbatively and so are crucial in describing theories globally. We review the framework of topological wormholes in quantum gravity and their relation to holographic entanglement. Thereafter, we initiate a study on the entanglement structure of qubits in potential-well lattices with topological wormholes. In these lattices we demonstrate the relationship between quantum tunneling events and wormholes.

Contents

1 Introduction

A recent topic of interest in theoretical physics has been the proposition of *emergence*; is the macroscopic structure of our universe a consequence of its fine-structure at smaller length scales? In other words, are large scale system behaviours dependent on its constituent parts, or is it independent? These questions have garnered significant attention when describing the nature of our spacetime, and whether or not it possesses some deeper structure as a result of emergence.

The study of spacetime structure is a topic of great interest in the realm of quantum gravity. It can be approached from two distinct perspectives: quantizing classical gravitational theories and investigating theories where quantum effects and spacetime emerge as fundamental properties. The former involves methods like loop quantum gravity, which seeks to capture the discrete nature of spacetime at microscopic scales. On the other hand, the latter explores the intriguing possibility of the universe's geometry emerging through the interplay of fundamental processes, such as phase transitions driven by the renormalization group flow of the theory's coupling constants [1]. Further studies of spacetime emergence have come from the study of entanglement in black holes. Black holes pose interesting contradictions, such as the *black hole information paradox* whereby information input into black hole is lost since what is radiated by the black hole is featureless thermal radiation [2]. This means at some point the system loses information, which violates the conservation of information in an isolated system (our universe). To reconcile this inconsistency, it was proposed that the radiation emitted is entangled with the interior of the black hole [3], meaning that the information is still present in the system globally and there is no violation in the amount of information. The system is described by its *entanglement entropy* which describes the degree of which properties of objects are connected [4]. This can also be viewed as how much information is lost when the system is viewed partially. This picture later came with issues as when considering these black holes in the presence of external matter. As matter falls into the black hole — being that it eventually will be evaporated out from the black hole as radiation — the in-falling matter must be entangled with the radiation. This is problematic as the radiation is already maximally entangled with the inside of the black hole and this violates the monogamy of entanglement; maximal quantum entanglement cannot be shared amongst an arbitrary amount of parties. The reconciliation to this was that monogamy would not be violated if the radiation was *itself* also the interior of the black hole [5]. In this sense the radiation is not truly independent from its source, but is instead tethered to an *island* within the black hole. This presents an interesting example of emergence: the thermal radiation corresponds directly to a piece of space within the black hole. The features of the island emerges by the properties of its entangled radiation elsewhere. What is more is that the implication that the radiation is also the inside of the black hole seems to be contradictory being that they are physically separated in spacetime. However, the spatial separation implies a connection between the two in the form of a non-local extremal geometry: a *wormhole*. We will see further in this introduction that wormholes will be a center focus in probing the global properties of systems in quantum gravity.

The characterization of spacetime as an emergent property can be extended with the use of holography, specifically the AdS/CFT correspondence. This is a conjecture that relates the gravitational partition function in the bulk of the anti de Sitter (AdS) spacetime, to the conformal field theory (CFT) partition function on the conformal boundary of AdS. From this context we can interpret gravitational objects as fields on the boundary. This motivates the idea that the structure of spacetime bulk is an emergent property of a lower dimensional quantum system on the boundary [6], which is the principle of *holography*. A popular example is that pairs entangled particles on the boundary correspond to a wormholes in the bulk [5]. The idea of spacetime emergence from quantum mechanics was extended further in [7] in which pairs of entangled particles corresponds regions of the AdS space. A given amount of entanglement entropy corresponds to certain configuration of the bulk spacetime, which dictates the energy distribution within that space [8]. Furthermore, the entanglement entropy of the quantum particles could also in principle describe the entanglement entropy of matter in the bulk [9], or even endow spacetime with features that match those of general relativity [10].

Emergence of spatial structure is not unique to spacetime however, and also includes configurations of qubits and symplectic manifolds. A configuration of five highly entangled qubits can be used to store the information of a single virtual qubit. This construction is useful in the construction of quantum computers, as if a subset of the physical qubits are damaged, their entanglement makes it so that the information of the virtual qubit is not lost. In this sense we can think of this configuration as a model of emergent space [11]. The virtual qubit emerges as an atom of space from the entanglement structure of the qubits. Additionally, the entanglement structure of qubits turns out to be interesting as they describe different topological wormholes, which are identified by different geometric phases [12]. While our preceding discussions were of spacetime wormholes, *topological wormholes* are extremal geometries arising in the quantized phase spaces of quantum systems [13], which connect different independent system *orbits*. It was shown that topological wormholes occur in a large class of quantum mechanical systems, and not just holographic ones. Much like how there is a correspondence between entangled states and spacetime wormholes, it was also demonstrated that there is a connection between the topological wormhole partition function and the entanglement entropy of quantum states prepared in Euclidean spacetimes [14]. Wormholes are thus integral to describing the global structure of a system, whether it is a spacetime or symplectic manifold. We use the topological wormhole framework to describe entangled systems of qubits in potential-well lattices, and within it relate wormholes to quantum tunneling.

This paper is structured as follows. In section 2 we look at wormholes occurring in quantum gravity. In section 3 we look at bipartite qubit systems subject to potential-wells which admit topological wormholes and relate them to qubit quantum tunnelling events and in section 4 we summarize. Finally, the appendix A describes the framework for the topological structure of wormholes and their correspondences to entanglement with topological quantum field theory.

2 Wormholes in Quantum Gravity

Now we turn our attention to wormhole cobordisms occurring in quantum gravity. First we present a concise overview of spacetime wormholes and their dual entangled states in the context of AdS/CFT. Then we look at when the systems space or bundle admits non-trivial holonomy and how it gives rise to hidden information. Finally, we look topological wormholes which manifest in quantized symplectic spaces.

2.1 Spacetime Wormholes & Holography

Here we look at the spacetime framework of eternal black holes in the context of holography. The eternal black hole spacetime, also known as the *maximally extended Schwarzschild black hole*, has its boundary and singularity identified in multiple universes and exists for all time (even before the big bang in inflationary models). This is due to the spacetime not admitting a global time-like killing vector being that the different definitions of time flow on the boundaries of the black hole generate a topological defect at the event horizon. A constant time slice of the spacetime geometry leads to a hypersurface that connects the spaces of the two universes, a geometry known as a *wormhole* or an *Einstein-Rosen* *bridge*. The interpretation is that the interior of the internal blackhole is the bulk of the wormhole; to traverse the wormhole you must enter the blackhole. The spaces which the wormhole connects are cobordant if have the same dimension and their disjoint union is the boundary of a compact manifold which is one dimension higher (the 3+1D spacetime slices). In general these wormholes are non-traversable as the wormhole throat shrinks in size as an observer enters the eternal black hole's horizon in their universe and approaches the singularity [5]. What's more is that the wormhole geometry grows with the expansion of the universe, so even if the opening radius is fixed, an observer would be stuck inside the wormhole once the throat ends become causally disconnected. However, it was found that if an interaction is turned on that couples the two boundaries[†], the quantum-matter stress tensor ends up having negative average energy and this prevents the throat opening from closing [15]. After the gravitational field of the wormhole interacts with the background spacetime (gravitational backreaction), the wormhole is rendered traversable. Furthermore, the study also revealed that infinite null geodesics which enter the wormhole must be chronal (sets of points are chronal if any two points can be connected by a timelike curve) and so wormholes *cannot* be used to violate causality or for faster than light travel.

These wormholes are particularly interesting when the spacetimes they connect are anti-de Sitter space (AdS). This is because of the AdS/CFT correspondence, a conjecture which proposes a duality between gravitational theories on the bulk with conformal field theories on the conformal boundary in the form of related partition functions [16]. To illustrate this, consider a massless scalar field ϕ in the bulk of an $(n+1)$-dimensional anti-de Sitter space, AdS_{n+1}. The restriction ϕ to the conformal boundary is denoted as ϕ_0 and is coupled to a conformal field $\mathcal{O}$ (a field that is invariant under the conformal symmetry group) under the coupling $\int_{S^n} \phi_0 \mathcal{O}$, where S^n is the conformal boundary of AdS_{n+1}. We can relate the supergravity partition function Z_S on the bulk associated with ϕ, with the conformal field partition function on the boundary as:

$$Z_S[\phi_0] = \left\langle \exp \int_{S^n} \phi_0 \mathcal{O} \right\rangle_{\mathrm{CFT}} = Z_{\mathrm{CFT}}[\phi_0]. \tag{2.1}$$

This duality allows us to identify gravitational objects in the bulk to conformal fields on the boundary of AdS. Perhaps the most popular correspondence comes from the ER=EPR conjecture which states that wormholes in the spacetime bulk are dual to quantum entangled states living in the CFT on the boundaries. For an external black hole that connects two AdS spaces (where they are denoted as the left and right AdS spaces), we have two copies of CFTs that live on the boundaries of the two connected spacetimes. Such a wormhole connecting the two spacetimes corresponds to an entangled state that live in both CFTs. The entangled state between two identical CFTs is known as the *thermofield double state* (TFD) and is written as:

$$|\Omega\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} \, |n\rangle \otimes |n\rangle^* \,, \tag{2.2}$$

[†]The interaction comes from a deformation of the theory's action with an extra term in the form of a product of two single trace operators (a trace of a matrix product of field operators).

where $Z(\beta) = \mathrm{tr}(e^{\beta H})$ is the thermal partition function of the state for a fixed inverse-temperature β and Hamiltonian H, E_n are the energy eigenvalues of the Hamiltonian, and $|n\rangle^*$ is the CPT conjugate of $|n\rangle$ to account for time flow running in different directions on the conformal boundaries. The energy eigenstates live in the Hilbert vector space $\mathcal{H}_L \otimes \mathcal{H}_R$ over the boundary, where $(\mathcal{H}_L, \mathcal{H}_R)$ are the Hilbert spaces of the left and right CFTs, respectively. The state is prepared via path integrals in the Euclidean AdS spacetime, AdS_E, which is represented as boundary conditions on hypersurfaces of AdS_E [17]. The state is then evolved in time in the usual Lorentzian AdS space. This represents an *entangled state* as it cannot be factored out into a single product state. The holographic duality gives us a correspondence between wormholes and entanglement.

Finally, spacetime wormholes give extra contributions to the Witten diagrams (roughly speaking they are Feynman diagrams projected onto the Poincaré disks of AdS space) when computing correlations functions in the CFT [18]. This develops in the form of extra terms known as defects, which connect different parts of the conformal diagram via wormholes. This allows for the boundary operators to evolve in different ways, such through the throats of the wormhole. In the following section we look at what happens when the systems space or bundle admits a non-trivial holonomy.

2.2 Hidden Information in Holonomy

There is more to the story when the manifold admits non-trivial holonomy, which in essence captures the non-commutative nature of the space we are working with. If the space is in fact a *fibre bundle* then the holonomy measures how much the endpoints of a closed path in the base space differ when uplifting the path to the total space of fibres [19]. In the case of the eternal black hole, the non-trivial holonomy arises from the topological defect at the event horizon which causes a discontinuity in uplifted time-like Killing paths. This leads to wormhole contributions in the gravitational path integral due to a non-exact symplectic form [20], which will be discussed in the following section. What's more is that in physics then holonomy is often referred to as a *geometric phase* and is attributed to hidden information within a quantum system. This is because manifolds admitting a non-trivial holonomy require multiple coordinate patches and locally a physical observer only perceives one patch of the base manifold. This means that they cannot know that the system admits a geometric phase defined via a path that goes through multiple patches. This phase tells us whether or not the structure is a product space or a fibre bundle and so this information is hidden from a local observer [19]. In the following we look at the consequence of geometric phases and how they distinguish between entangled states with the same entanglement entropy.

Consider the conformal spacetime geometry of an eternal blackhole, consisting of two AdS spacetimes (and their respective conformal boundaries), and the interiors of the black and white holes. If we foliate the spacetime by constant-time hypersurfaces (as in figure 5), the leaf associated with $t = 0$ corresponds to the usual wormhole geometry connecting two spatial regions of the AdS spaces. A geodesic on this geometry is holographically dual to thermofield double state $|\Omega\rangle$ living in the Hilbert space over the conformal boundaries of the leaf. Although the Hilbert space on the boundary factorizes into a tensor product of the Hilbert spaces at each assymptotic boundary, we cannot factorize the bulk Hilbert space over each leaf due to the presence of wormhole structures [19], which is the same reason why we cannot factorize of the wormhole partition function globally. Leaves with $t > 0$ instead correspond to time-shifted wormhole geometries, and the geodesics within those geometries are dual to TFD-like states that have additional phases [12]:

$$|\Omega^{\alpha}\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{i\alpha_n} e^{-\beta E_n/2} |n\rangle \otimes |n\rangle^* \,, \tag{2.3}$$

where α_n are phases that distinguish the different entangled states and hence distinguish different wormhole leaves. These states are interpreted as the microstates of the eternal black hole [21]. The time-shifted wormholes have the same geometry but have different identifications of boundary times. One musts ask where exactly do these phases come from? The spacetime manifold is not necessarily connected and so must be described by different coordinate charts. Consider the left and right copies of AdS to have the respective coordinate charts (ε_L, L, t_L) and (ε_R, R, t_R). Here ε_L is the asymptotically flat region around the left wormhole throat, L is the boundary of the left AdS space, and t_L is the time coordinate flowing along the left boundary. These charts overlap in the interface of the asymptotic regions around the throats, and at the wormhole horizon (interface of the interior of the wormhole where the left and right throats connect). In other to keep the spacetime manifold well defined, we must make it so that the transition functions between the charts are smooth, resulting in the relations: at the interface of the asymptotic regions $t_L = t_R$, and at the interface at the horizon $t_L = 2\delta - t_R$, where δ is a fundamental degree of freedom of the system [22] (which is actually an element of the system's moduli space; more on this in the qubit section). With this parameter the phases can be computed as $\alpha_n = -2E_n\delta$.

What happens when we transport a particle around a closed geodesic on one of these wormhole leaves? To transport it we require groups that generate symmetries and so evolve the states. Consider a *Lie symmetry group* G which also has the structure of a manifold[†]. We will need a connection if we wish to describe derivatives covariantly and preserve group symmetries. This can be represented as the *Maurer-Cartan* form $\Gamma = g^{-1}dg$ which is the natural connection on a group manifold, where $g \in G$ and $dg \in T^*_g G$. This form carries the basic infinitesimal information about the symmetries and transformations associated with G. Moreover, the tangent space at the identity of the group, $T_{\mathbf{1}}G$, is the vector space over G known as the *Lie Algebra* $\mathfrak{g}$. This consists of generators which encode the dynamics of quantum states in spacetime via the exponentiation of the Lie algebra elements.

With this we move onto parameter spaces, which is the space of external parameters that define the system's Hamiltonian. For example, these parameters could be the components of an external gauge field as in the Aharonov-Bohm effect. When a system evolves adiabatically, the parameters vary slowly such that the system stays in an eigenstate of the Hamiltonian. When the system returns to its initial state after completing a cyclic evolution in parameter space, the quantum state picks up a non-trivial phase known as a

[†]This can be extended in generality by having a fibre bundle of group manifolds in which the connection endows a covariant derivative on the fibers of the group bundle.

geometric phase (more commonly known as the *Berry phase*) which is an observable property of the system. This evolution corresponds to parallel transporting a quantum state along a closed loop in parameter space. It is noted that in the case of the Aharonov-Bohm effect the charged particle moves in a loop in physical space which *corresponds* to a closed parallel transport loop in the parameter space which ultimately gives the particle state the phase. Explicitly, the particle moving around the solenoid evolves the system in such a way that a closed loop path is followed in parameter space, and it is **coincidental** that physically the particle also moves in a loop. This is analogous to the case of wormholes where transporting a pair of entangled particles through the wormhole throats in a closed loop happens to change the system's Hamiltonian to go around a loop in parameter space and give us a Berry phase. Being that the geodesic that generates the Berry phase is dual to an entangled state, we must evolve both particles via some *unitary* group elements $U_a \in G$ to preserve probability amplitude normalizations. Moreover, what if one of the entangled particle say interacts with an external field? This gives us distribution of entangled states by deforming the unitary operators of one of the Hilbert subspaces. We define the unitary group representation $U = U_L \otimes U_R$, where $U_{L/R}$ acts on the left/right CFT Hilbert space. We can consider a continuous parameter $\lambda \in [0, 1]$ to deform U_R such that $U_R = \mathbb{1}$ for $\lambda = 0$ and $U_R = U_L$ for $\lambda = 1$ [12]. This gives us a continuous spectrum of entangled states for different values of λ which have the same entanglement entropy (the unitary operators can be moved around in the trace functional and cancel in the definition of the entanglement entropy).

Finally, to describe the Berry phase we must make use of the Berry connection, which is defined in terms of the previously mentioned Maurer-Cartan form $\Gamma = U^{-1}dU$ for a group element $U = U(\lambda)$ that distinguishes different entangled states for different λ. Given the form Γ, we can define the *Berry connection* for a given state which we want to consider (the phase-shifted TFD state $|\Omega^\alpha\rangle$) as:

$$A = i \langle\Omega^\alpha| \Gamma |\Omega^\alpha\rangle \,. \tag{2.4}$$

Here the Berry phase is the holonomy of the Berry connection. To compute the phase we must define the *Berry curvature* form F which is given by $F = i \langle\Omega^\alpha| dA |\Omega^\alpha\rangle$, where dA is the associated symplectic form of the parameter space which is not globally exact. The *Berry phase* Φ is then given integrating the curvature over a closed loop in parameter space γ:

$$\Phi = \oint_\gamma F(\lambda). \tag{2.5}$$

As long as $\lambda \neq 1$ so that $U_L \neq U_R$, the system admits a non-zero Berry phase. Using the inverse function theorem for a well-behaved function $\Phi(\lambda)$, we can invert this relation to get $\lambda(\Phi)$. Thus, the Berry phases distinguish a class of entangled states with the same entanglement entropy/structure. This corresponds to transporting particles along closed loop geodesics on the different time-shifted wormhole leaf geometries. Furthermore, the class of states with the same entanglement entropy but different Berry phases is a manifestation of the non-factorization of the leaf Hilbert spaces [12], and that the symplectic form cannot be globally exact. Finally, there are different types of Berry phases in holographic CFTs which are classified by the type of bulk diffeomorphisms that are involved [23].

2.3 Topological Wormholes in Symplectic Spaces

We now consider topological wormholes arising in theories of quantum mechanics. We look at how we can arrive at these geometries by quantizing a classical symplectic space via geometric quantization, and looking at orbits occurring in it via symplectic reduction. We first look at quantizing a phase space, and then look at it in the context of wormholes.

2.3.1 Geometric Quantization

Let's start with classical mechanics. Consider a $2n$-dimensional phase space Σ with coordinates $\sigma^a = (q^1, \ldots, q^n, p^1, \ldots, p^n)$, where q^a are the generalized position coordinates, and p^a are the generalized momentum coordinates. Much like how spacetime is endowed with a symmetric bilinear two-form $g_{\mu\nu}$ – the metric tensor – phase spaces are symplectic manifolds which are endowed with a non-vanishing antisymmetric two-form:

$$\Omega = \frac{1}{2}\Omega_{ab} d\sigma^a \wedge d\sigma^b. \tag{2.6}$$

Here Ω_{ab} are the components of the two form, represented in the cotangent bundle basis $d\sigma^a$. Being that Ω vanishes nowhere on Σ — meaning it is non-degenerate — Ω_{ab} has an inverse. The symplectic form encodes the dynamics of the classical system. To see this, consider two functions $f, g \in C^\infty(\Sigma)$, the Poisson bracket of them takes the form [24]:

$$\{f, g\} = \Omega_{ab} \frac{\partial f}{\partial \sigma^a} \frac{\partial g}{\partial \sigma^b}. \tag{2.7}$$

Moreover, we can use this definition of the Poisson bracket to evolve functions in time in the following form:

$$\dot{f}(t, \sigma^a) = \left(\frac{\partial}{\partial t} - \{H, \cdot\} \right) f, \tag{2.8}$$

where H is the Hamiltonian of the system which describes time flow of the system over the symplectic space, and $\{H, \cdot\} f = \{H, f\}$. The Poisson bracket is closely related to the Lie derivative via $\{f, g\} = \mathcal{L}_{X_g} f$ (where X_g is the vector field associated with the function g) and so it gives us information about symmetries and conserved quantities of the system.

Thereafter, we can consider a coordinate system of the phase space which diagonalizes the Hamiltonian, which are called *action angle coordinates*. This allows us to study the normal modes of a system without having to solve the equations of motion. This is a canonical transformation from (q^a, p^a) to (J^m, θ^m) which preserve the structure of the equations of motion. Here J^m are the *action coordinates* given by $J^m = \oint dq^m p^m$ which are conserved quantities relating to the energy of the system and the amplitude of the oscillation modes. θ^m are *angle coordinates* which are the canonical conjugates to J^m, which are related to the phases of the oscillation modes. The intuition for these coordinates can be found when considering orbits. The set of all trajectories of a subsystem in phase space given by the action of a symmetry group are known as *orbits*. These orbits represent the evolution of the system and are usually parametrized by time. Along these orbits J^m is conserved, so the interpretation is that J^m label different orbits in phase space that the system can evolve through and θ^m can parametrize these orbits.

Now, instead of diagonalizing the Hamiltonian via a canonical transformation, we can instead consider an unperturbed diagonal Hamiltonian which is a function of the action angle coordinates $H_0(J)$, in the presence of a perturbing Hamiltonian which is a function of the phase space coordinates $H'(\sigma)$ such that $H' \ll 1$. This gives us a Hamiltonian of the form $H = H_0(J) + H'(\sigma)$ [14] (where H_0 not a function of θ being that its canonical conjugate is conserved) and this allows us to use the framework of perturbation theory. This Hamiltonian results in the symplectic form picking up an extra term:

$$\Omega = \Omega_{ab} d\sigma^a \wedge d\sigma^b + \delta_{mn} dJ^m \wedge d\theta^n, \tag{2.9}$$

where δ_{mn} is the usual Kronecker delta matrix. Moving forward, we want to quantize the phase space so that we can consider a quantum theory. This will be done via geometric quantization, but alternatively deformation quantization works as well. We will present a brief overview of the procedure using natural units, but a rigorous treatment of it can be found in [25]. The procedure is as follows:

We want to lift classical observables from our symplectic manifold Σ to quantum operators in some Hilbert space $\mathcal{H}$ in a way that preserves the algebraic structure of Σ. The first step is known as *pre-quantization* in which we define a pre-quantum Hilbert space $\tilde{\mathcal{H}}$. We begin by defining a line bundle L (a vector bundle where the fibres are one-dimensional vector spaces) L over Σ which is equipped with a $U(1)$-connection such that the curvature form is $i\Omega$. Explicitly, we say the symplectic form is the curvature form of a $U(1)$-principle bundle written as the fibration $L \to \Sigma$. This bundle is called the *pre-quantum line bundle* [25]. This construction requires that Ω obeys the Bohr-Sommerfeld condition which states that $\Omega/2\pi$ forms an integral cohomology class. Essentially this means that integrals of $\Omega/2\pi$ over cycles of Σ must be integers. We define the *pre-quantum Hilbert space* $\tilde{\mathcal{H}}$ as the collection of square-integrable sections of L. We can now begin our construction of pre-quantum operators.

For a classical observable on the phase space given by a smooth function $f \in C^\infty(\Sigma)$, the associated pre-quantum operator is the linear map mapping:

$$Q(f) : \Gamma(L) \to \Gamma(L). \tag{2.10}$$

Here $\Gamma(L)$ is the space of smooth sections of the line bundle, whose elements are the pre-quantum states. If we select $\psi \in \Gamma(L)$, the map Q acts on it in the following way [26]:

$$\psi \mapsto -i\nabla_{v_f}\psi + f \cdot \psi, \tag{2.11}$$

where ∇_{v_f} is the covariant derivative of sections along v_f which is specified by the bundle connection, and v_f is a Hamiltonian vector field[†] corresponding to the function f. The pre-quantum operators satisfy the following commutator algebra:

[†]A vector field is *Hamiltonian* if the flow is generates on Σ describes the time evolution of states.

$$[Q(f), Q(g)] = iQ(\{f, g\}), \tag{2.12}$$

where $[,]$ are the usual commutator brackets associated with the Lie derivative.

We are ready to move onto the next step of geometric quantization (ignoring the metaplectic correction for non-trivial topologies): *polarization.* In essence, the pre-quantum Hilbert space $\tilde{\mathcal{H}}$ is too big in the sense that phase space is much larger than the physical configuration space. So what we do is carefully select a subspace of the pre-quantum Hilbert space $\tilde{\mathcal{H}}$ such that we eliminate redundancies and ensure that the resulting quantum theory captures the relevant degrees of freedom. First, consider the tangent bundle $T\Sigma$ associated with the symplectic phase space Σ. To capture the complex degrees of freedom present in quantum mechanics we complexify the tangent bundle by endowing it with a complex structure given by the two-tensor J^i_j such that $J^2 = -\mathbb{1}$. A *polarization* is a choice of a Lagrangian subbundle of the complexified tangent bundle $T\Sigma^{\mathbb{C}}$, or rather at each point in the complexified tangent space we select a Lagrangian subspace. A subspace is *Lagrangian* if it is isoptropic and is half the dimension of the space it is a subspace of. In this sense these subspaces of the tangent bundle on Σ form an integral distribution of subspaces which foliate Σ. With this we can define the *quantum Hilbert space* $\mathcal{H}$ to be the space of all square-integrable sections of L that are covariantly constant in the direction of the polarization. This might be mysterious at first glance, but the sections are vector fields which don't change orientation when parallel transported about the polarization subspace. This condition ensures that resulting quantum states are compatible with the classical and quantum symmetries of the system. Using the quantum Hilbert space, our pre-quantum operators become quantum operators in the theory $\hat{Q}$ which act on elements of the Hilbert space $\mathcal{H}$. Now that we have a notion of quantizing a classical phase space to produce a quantum theory, we move onto looking at this in the context of topological wormholes.

2.3.2 Symplectic Reduction

Now we can finally look at topological wormholes occurring in quantum theories. For a generic theory of classical mechanics, we can write the partition function of the theory at a fixed inverse temperature β as the following Euclidean path integral representation:

$$Z(\beta) = \int [d\sigma] \exp\left\{ \int_D \Omega - \int_{\partial D} dt\ H \right\}, \tag{2.13}$$

where $[d\sigma]$ is the usual product path integral measure associated with phase space coordinates σ^a, Ω is the symplectic two-form, H is the Hamiltonian, and $D \subset \Sigma$ is a two-dimensional submanifold of the symplectic phase space Σ. The argument of the exponential is the action of the system, and the perturbative expansion of the gravitational path integral includes contributions arising from wormhole structures. Now, being that Ω is closed such that $d\Omega = 0$, its integral over D is a topological invariant and so the geometric properties of D don't matter, only its topology [14]. With this we can select the most trivial topology for D, taking the form of a 2-disk which is embedded in Σ. The boundary of this disk is an *orbit*, which is the space of all trajectories through which a state can evolve in phase space that represents the time evolution of a system. Because of this is it natural to parametrize the boundary of D with time t and it represents independent orbit of a particle. One could imagine connecting two disks between two independent particle orbits which otherwise would never intersect, via a minimal surface area hypersurface, a wormhole. In this sense, the *topological wormhole* connects independent particle orbits in phase space which can be extended to connecting n-particle orbits, which forms the *n-fold trumpet geometry.* This is identical in topology to the *n-fold replica wormhole geometry* which are spacetime wormholes that connect n spacetimes of the same dimension. It is noted that thus far this subsection has been classical.

What about the quantum operators associated with the wormhole on these phase space subspaces? The last step we have to do is known as *symplectic reduction*, a process which involves looking at a subregion of the symplectic space Σ (an orbit) while preserving the structure and symmetries of the system. The Guillemin-Sternberg geometric quantization conjecture (proven in [27]) states that the order of geometric quantization and of symplectic reduction can be interchanged without changing the result, so we chose to reduce and then quantize. This is to simplify calculations which might lead to non-transcendental solutions. The procedure is as follows. As is usual in quantum theories, consider a connected Lie group G which acts on a manifold, which in this case is our symplectic manifold (Σ, Ω) with a symplectic form/structure Ω. The action of a Lie group is said to be *Hamiltonian* when the elements of its associated Lie algebra, denoted as $\mathfrak{g}$, possess associated vector fields that are Hamiltonian. The Hamiltonian action of the lie group on Σ is equivalently described by the *moment map* μ:

$$\mu : \Sigma \longrightarrow \mathfrak{g}^*, \tag{2.14}$$

where $\mathfrak{g}^*$ is the *dual Lie algebra* (much like how the tangent bundle is dual to the cotangent bundle). We consider the identity of $\mathfrak{g}^*$ being $0 \in \mathfrak{g}^*$ and consider the submanifold this corresponds to in Σ given the inverse map $\mu^{-1}(0) = M \subset \Sigma$. The logic behind selecting this region is similar in principle to working near the identity of the lie algebra to generate its elements via an exponential map; there is enough information within M that we don't need to consider the full symplectic space Σ. If G acts freely on M (it has no non-trivial fixed points) then it turns out that if the quotient space M/G is a smooth manifold, then it inherits a non-trivial symplectic structure. Thus we say that M/G is the *symplectic reduction* of Σ that inherits a unique symplectic form ω whose pullback from Σ to M is exactly the restriction of Ω to M. In this case we call M/G the *orbit* of the system which is the set of elements in M that can be moved by elements of G. On these orbits the symplectic form is **not** exact is an indicator of the presence of topological wormholes. This relates direct to non-factorization of the partition function of topological wormhole (we will see in a bit it is also an indicator of entanglement).

Now that we have selected an orbit for the system which will reduce calculation complexity, we quantize it using geometric quantization as we covered before. From the classical observables of the system we can construct the set of quantum operators acting on the Hilbert space as [14]: $\mathcal{O}(\sigma, \theta) = J^m(\sigma), \sigma^a, W(\theta)$. Here σ^a commute with the unperturbed Hamiltonian H_0 as such are *Noether currents*, J^m are *Casimir operators* (such that $J^2 = \sigma^a \sigma_a$), and $W(\theta)$ are *open Wilson lines*. Similar to usual spin eigenkets $|j, m\rangle$ in quantum mechanics, here we have eigenkets $|j, m, s\rangle$ associated with the eigenvalues of the operators $\mathcal{O}(\sigma, \theta)$, where the extra quantum number s comes from a degeneracy of irreducible representations of the σ operator algebra with the same value of the Casimir eigenvalue. The different quantum numbers (j, m, s) correspond to whether or not the orbits that are connected by the topological wormholes are classically correlated, quantum entangled, or classically uncorrelated [13]. It is noted that the different values of the Casimir eigenvalues corresponds to different leaves of the foliation of the complexified tangent bundle when performing a geometric quantization of the orbit.

Finally, much like how there is a correspondence between spacetime wormholes and entangled states when looking at AdS spaces, here we have a correspondence in the form of the partition function of the n-fold topological wormhole is identical to the n-th Rényi entropy (entanglement entropy) of a thermo-mixed double state. A *thermo-mixed double state* is similar to the thermofield double state, with the exception that the two CFTs need not be identical, and that the two systems the wormhole connects are different inverse temperatures β. In this sense spacetime wormholes connect different universes and topological wormholes connect different orbits, both which are dual to entangled thermofield double states. In the following work we look at applying this formalism to qubit systems admitting topological wormholes which are subject to arbitrary well potentials.

3 Wormholes in Qubit Networks

Now we look at topological wormholes in the context of qubit systems, and how its entanglement structure may be modified with an arbitrary distribution of potential wells.

3.1 Entanglement Structure of Qubits

There is a lot of mention of how the non-exactness of the symplectic form Ω gives rise to wormhole geometries, but it is not very intuitive. We can have a clearer picture of this manifestation in the context of qubit systems. We will also see how to further characterize different entangled states through different orbits.

In classical computers, the infinitesimal information required for computations are given by *bits* which take a value from the set $\{0, 1\}$. On the other hand, *qubits* are an extension of this where we can have some state $|\psi\rangle$ that it is a superposition of these binary values, given by $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ for some coefficients $\alpha, \beta \in \mathbb{C}$. The advantages of using such a construction for computations in a quantum computer is the non-localization of information through entanglement. For a system admitting n-entangled qubits, if a subset of the system is corrupted, the information is still globally preserved. This is characterized by the entanglement entropy which tells you how much information is lost when looking only at a subset of the system.

Now, to preserve unitarity of the system we require that the state evolves under unitary operators which make the norms of quantum states invariant. Consider a basis in $\mathbb{R}^3$, where on one of the axes we associate the positive direction to be the state $|0\rangle$ and the negative direction to be $|1\rangle$. The state $|\psi\rangle$ is some vector represented in $\mathbb{R}^3$ in the qubit basis where its direction dictated by α, β and has unity norm. We can consider the action of a unitary symmetry group on the state which rotates it, and keeps it in a superposition of $|0\rangle$ and $|1\rangle$. If we apply all possible unitary transformations, this traces out a unit 2-sphere which represents all possible states of $|\psi\rangle$: its Hilbert space $\mathcal{H}$. This is known as the *Bloch sphere* and looks like the following:

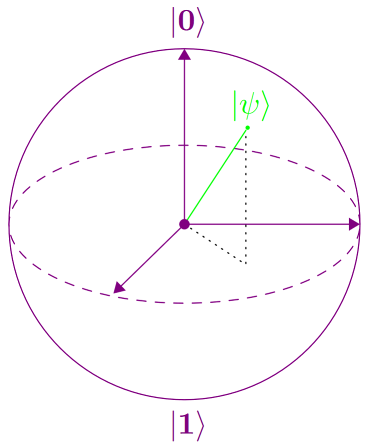

Figure 1. Representation of a spin-1/2 particle with the Bloch sphere in three dimensions.

The Bloch sphere is a Hilbert space for a single particle with spin, and is described by the complex projective space $\mathbb{C}\mathrm{P}^1$ with coordinates z_i. The associated spin operators of the Bloch representation are $S_a = (1/2) z_i^* \sigma_a^{ij} z_j$, where σ_a are the usual spin-1/2 Pauli matrices [12]. For a system of two qubits (which we take to be entangled) with no interaction or external fields, the system is represented as $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$. This can be embedded as diagonal blocks in $\mathbb{C}\mathrm{P}^3$ and explain the local properties of the system. However, for *non-local correlations* such as those that arise from wormhole geometries, the embedding is no longer diagonal and the full $\mathbb{C}\mathrm{P}^3$ system must be considered. This is analogous to spacetime metrics no longer being diagonal when some external field is turned on and it breaks some of the spacetime isometries. This gives us a new interpretation of the condition of the symplectic form: globally we cannot make the symplectic form exact nor diagonally embed $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ into $\mathbb{C}\mathrm{P}^3$ due to non-local contributions arising from wormhole geometries. However, locally these contributions are not present and so we can construct the symplectic form to be exact and diagonally embed the two-qubit system. In this case, much like the presence of a non-exact symplectic form, the non-diagonal embedding means the presence of topological wormholes in the phase space of the quantum system.

What about the entanglement structure of these pairs of qubits? Let's consider the case where an entangled pair of qubits lives on the conformal boundaries of the eternal black hole. To look at its entanglement structure we must consider moduli spaces with orbits fibred over them. In quantum gravity, the *moduli space* is the space of all possible solutions or configurations of a physical system. An example of such is the space of all possible metrics endowed on the internal space to be compactified in string theory. Now, for each asymptotic boundary (left and right) we consider a subset of diffeomorphisms that leave the conformal boundary conditions invariant, given by $G_{L/R}$. This gives us a total asymptotic symmetry group: $G_L \times G_R$. Moreover, the set of diffeomorphisms that leave the spacetime bulk information invariant is given by the diagonal subgroup of $G_L \times G_R$ and is given by G_D [19]. Thus, we define the moduli space $\mathcal{G}$ of the system as the quotient of these groups:

$$\mathcal{G} = \frac{G_L \times G_R}{G_D}. \tag{3.1}$$

The moduli space contains parameters or degrees of freedom[†] which fix the particular bulk spacetime solution. Thereafter, we consider a fibre bundle construction where the base space is the moduli space previously defined, and the total space is the union of all Hilbert space fibres over each point in the moduli space. Following the description of geometric quantization in the previous sections, the fully quantized quantum Hilbert space is the set of all sections of the bundle. *How do we define the Hilbert spaces at each fibre?* For the general case of qubits, instead of considering the projective Hilbert space $\mathbb{C}\mathrm{P}^3$, we instead take submanifolds of $\mathbb{C}\mathrm{P}^{n^2-1}$ for an $(n \times n)$-dimensional bipartite quantum theory (both qubit states have their own n-dimensional Hilbert space) [19]. Like in the case in section 2.3, our space is too large for the configuration space and so we have to select submanifolds of it. The submanifolds of $\mathcal{H}$ are *orbits* which are quotient spaces that are associated to different values of entanglement. These are constructed by quotienting local unitary transformations of the bipartite system — described by $U(n) \times U(n)$ — by symmetries of the entangled state for a given value of entanglement entropy. The quotient space subspaces are known as *entanglement orbits* which describe all possible states of the system. These are closely related to the symplectic orbits in the topological wormholes section which describes all possible evolutions of the system. In either case the contribution to the wormhole partition function is still integrating the symplectic form Ω over an orbit. Much like the case with $\mathbb{C}\mathrm{P}^3$, there is no diagonal embedding of spaces within $\mathbb{C}\mathrm{P}^{n^2-1}$ due to non-local wormhole structures. Instead different orbits of $\mathbb{C}\mathrm{P}^{n^2-1}$ give rise to different entanglement orbits [19] for a given value of entanglement entropy. For example, a product state with vanishing entanglement entropy lives within the orbit $\mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}\mathrm{P}^{n-1}$. We note that the symplectic form Ω of $\mathbb{C}\mathrm{P}^{n^2-1}$ vanishes when restricted to this orbit and so does not give rise to any wormhole contributions to the gravitational path integral. On the other side of the spectrum, states with maximal entanglement entropy live within the orbit $\mathbb{1} \times \mathrm{SU}(n)/\mathbb{Z}_n$. Finally, there is a defined spectra of states that are between vanishing and maximal entanglement entropy which live in the orbit:

$$\frac{U(n)}{U(1)^n} \times \frac{SU(n)}{\mathbb{Z}_n}. \tag{3.2}$$

[†]Recall in the previous section that a time parameter δ emerged from the transition functions at the horizon of the black hole. This is in fact a *bulk degree of freedom* and is an element of $\mathcal{G}$.

The intermediate entanglement entropy orbits (and their volumes) are labelled by the external parameters of the theory's Hamiltonian. Both the orbits of maximally entangled and intermediately entangled states will contribute to the gravitational path integral in the form of integrating Ω over the orbits. Being that there are more states with vanishing entanglement entropy, the orbit of the null-entanglement entropy states is much larger (larger symplectic volume) than the orbit for say, the maximally entangled states. Moreover, it is possible define operators such that states can flow between different orbits [19].

Does this have any connection to the topological wormholes we saw in the previous section? Yes! The orbits of non-vanishing entanglement entropy are in fact Lagrangian submanifolds of the total projective Hilbert space $\mathbb{CP}^{n^2-1}$ each endowed with a non-vanishing curvature 2-form. This means these orbits are symplectic spaces (much like the orbits as a result of symplectic reduction) with symplectic forms Ω defined on them. Each of these orbits have a corresponding geometric phase Φ which is obtained via integrating the symplectic form over the orbit volume:

$$\Phi = \int \Omega = \int d\sigma^a \wedge d\sigma^b \, \Omega_{ab}, \tag{3.3}$$

where σ^a are the coordinates on the orbit. This is equivalent to the geometric phase calculated with the Berry curvature form in section 2.3, and contributes wormhole corrections to the gravitational partition function of the system. This is due to the orbits having non-trivial holonomy. Furthermore, being that this is an integral over the symplectic volume, it also characterises the number of states within a specific entanglement orbit [19].

Now that we have looked at the entanglement structure of qubit systems via entanglement orbits, we move on to including potential wells which modify the entanglement structure the system in the following section.

3.2 Entangled Qubits in Potential Wells

In the context of quantum gravity we have considered two independent systems allowed to interact via some extremal geometry. This manifests in the form of wormholes, whether it occurs in spacetime to connect independent universes or quantized symplectic spaces to connect independent particle orbits. Thus far these systems have been completely independent until connected via wormholes so it would be interesting to study a system which is initially only partially independent. For example one could consider particles in separate infinite wells which constitute independent quantum systems. However, if we let the potentials be finite then there is a probability of these two classically separate systems interacting via quantum tunneling. These tunneling events extend the possible dynamics of the particles and consequently contribute non-perturbatively to the partition function of the particle ensemble from which we derive the statistical properties of the system. We look at how this affects the entanglement structure of qubits in potential-well lattices and the relationship between the non-perturbative wormhole and tunneling contributions to the system's partition function. To demonstrate this we study a 1D system of two particles each with confined to a finite potential-well. This can be extended to an N-dimensional system with M particles in a straightforward manner, but this won't be covered here.

The evolution of this quantum system is governed by its Hamiltonian H but it is important to establish the formalism in which this arises. When we looked at topological wormholes connecting symplectic submanifolds — particle orbits — we constructed the quantum Hilbert space $\mathcal{H}$ as the collection of all square-integrable sections of a line bundle over the system's symplectic phase space. Thereafter, to describe the entanglement structure of a bipartite quantum system of qubits we looked at the bundle formed by the quantum Hilbert space $\mathcal{H}$ over the system's group moduli space $\mathcal{G}$. For a given point in the moduli space, certain submanifolds of the Hilbert space — entanglement orbits — gave us the minimum amount of information to describe the entanglement structure of the system. Identical to the case of particle orbits, these entanglement orbits could be connected by extremal geometries — wormholes — due to the non-exactness of the associated symplectic form. How then would we describe the dynamics of these particles moving through spacetime? This would be of what bundle to construct to extract the system's Hamiltonian.

To do so we consider a *Hilbert bundle* $\mathscr{H}$ as the fibration $\mathcal{H} \to \mathbb{R}$, where $\mathcal{H}$ is the quantum Hilbert space of the system and $\mathbb{R}$ is the domain of the proper time t which parametrizes the worldlines of the particles in its ambient spacetime $\mathcal{M}$. To allow for the inclusion of internal symmetries we define the *associated endomorphism bundle* $\text{End}(\mathcal{H}) = \mathcal{H} \otimes \mathcal{H}^*$ for some dual Hilbert space $\mathcal{H}^*$. Furthermore, to define an action we require differential forms to integrate over the spacetime. The set of all differential forms is given by the *exterior algebra* over the cotangent bundle of $T^*\mathbb{R}$, denoted as $\Omega T^*\mathbb{R}$. Combining the information of the internal symmetries and forms can be achieved via a tensor product between the two structures as $\text{End}(\mathcal{H}) \otimes \Omega T^*\mathbb{R}$. If we select some fibre $t \in \mathbb{R}$ in this bundle, we can impose a local trivialization to define the flat connection as the canonical 1-form A. This form is given by the *sections* Γ of its bundle and is defined as:

$$A = H dt \in \Gamma \left(\text{End}(\mathcal{H}) \otimes \Omega T^*\mathbb{R} \right) . \tag{3.4}$$

Here H is an endomorphism known as the *Hamiltonian operator*, and dt is the 1-form associated with the fibre point t. Because the form includes the information of the endomorphism, we say A is an *endomorphism-valued* differential form. With this we have the Hamiltonian to give us information on the dynamics of the system. The parallel transportation of A over the bundle corresponds to *time evolution* of the system and allows us to evolve the qubits in time.

For a 1D lattice with two sites — denoted by L and R for the left and right site — each with a particle confined by a well, we have the following Hamiltonian:

$$H = H_L + H_R = \left(\frac{p_L^2}{2m} + V_L(x) \right) + \left(\frac{p_R^2}{2m} + V_R(y) \right) , \tag{3.5}$$

where $H_{L/R}$ are the Hamiltonians of the left and right systems, $p_{L/R}$ are the momentum of the particles, $V_{L/R}$ are the potentials centered at the sites, m is the mass of each particle, and (x, y) are the trajectories of the two particles. The potentials are constructed in a way that their interface with other potentials is finite with a height of λ, but pieces of the potentials that are on the boundary of the lattice are infinite. This confines the particle to the bulk of the lattice where they are allowed to quantum tunnel. Using reflective boundary conditions coming from the asymmetric wells on the wavefunction of the particles, and for sufficiently small $k \ll \sqrt{2m\lambda}$, we get the energy levels of the system as:

$$E_{nk} = E_n^{(L)} + E_k^{(R)} = 2\pi^2\lambda \left[n^2 f^{-2}(\alpha - 0) + k^2 f^{-2}(\gamma - \beta)\right]. \tag{3.6}$$

Here λ is the height of the internal portion of the wells which separate the particles, (n, k) are the energy levels of the left and right particles, $(0, \alpha, \beta, \gamma)$ are the edges of the confining particle wells, and $f(\tau) \equiv \sqrt{2} + 2\tau\sqrt{m\lambda}$ is some 1-parameter function of the potentials. For this system we define the separable 2-particle state as as:

$$|\psi\rangle = \Big(|n\rangle \otimes |\sigma\rangle\Big) \otimes \Big(|k\rangle \otimes |\rho\rangle\Big). \tag{3.7}$$

Here $(|n\rangle, |k\rangle)$ are the energy eigenstates of the left and right systems, and $(|\sigma\rangle, |\rho\rangle)$ are the spin eigenstates for the different particles. For the study of the entanglement structure we can ignore the independent spin part of the state which leaves us with $|\psi\rangle = |n\rangle \otimes |k\rangle$. We could extend this and consider an *entangled state*, such as the thermofield double state (TFD) which admit a class of entanglement orbits for different entanglement entropies. To study the non-perturbative nature of quantum tunneling events, we first begin with a separate product state, and then move on to an entangled thermofield double state.

3.2.1 Separable States

For a pair of particles within a potential-well lattice of two sites, we construct their two particle state as a tensor product of their energy eigenstates. Such a state can be separated and so *does not* posses the property of entanglement. This means that the system lacks a non-trivial *entanglement structure* which efficiently stores information about the particles in a compact manner. Within the barriers of the lattice the particles can quantum tunnel through classically forbidden regions to interact with each other in different ways. Each one contributes to the system's partition function non-perturbatively such as in the form of an instanton. Of all different ways that the particles evolve with time, we consider the case where the particles swap positions and thus both tunnel through the barrier in opposite directions. For an initial particle state $|\Omega\rangle = |n\rangle \otimes |k\rangle^*$, and a final *swapped* particle state $|\mho\rangle = |k\rangle^* \otimes |n\rangle$, the transition amplitude K of this event is given by the following path integral:

$$K(\Omega, \mho) = \langle\mho| \, e^{-i(H_L + H_R)t} \, |\Omega\rangle = \int_{\Omega}^{\mho} \mathcal{D}x \, \mathcal{D}y \; e^{iS[x,y;E_{nk}]}, \tag{3.8}$$

where $S[x, y; E_{nk}]$ is the action functional, (H_L, H_R) are the Hamiltonian operators, and $\{\mathcal{D}x, \mathcal{D}y\}$ are the usual path integral product measures associated with the particle trajectories $\{x(t), y(t)\}$. In the case that classically the particles cannot pass through the well while evolving Lorentzian time, we analytically continue the temporal domain to evolve instead in imaginary Euclidean time to *quantum tunnel* through the barrier. This transition amplitude shows to be non-perturbative and will contribute to the system's partition function. Furthermore, the integrand is complex and so one must select contours in some complex plane to compute the integral. *Over what complex planes do we integrate this over and how do we select the corresponding contours in said planes?* For this we have to look a bit beyond the current analysis being used. Instead of considering particles tracing out a trajectory in the 1D space over time, we can instead consider them as moving along worldlines in some spacetime. For our case this could be some 1+1D spacetime $\mathcal{M}$ with non-trivial curvature and topology. To ensure closed form solutions we choose to foliate the spacetime into equivalent time slices (hypersurface leaves) or Cauchy surfaces $\mathcal{M}_t$, which are parametrized by some global time coordinate t. Being that the parametrization of time is allowed to be arbitrary, this causes some ambiguity in the definition of time flowing in the system. What we can do — as is done in the ADM formalism of quantum gravity — is impose reparametrization invariance via the symmetry $t \to N(t)t$ which gives rise to the following Lagrange multipliers [28]:

$$N(t) = -t^a n_a = \frac{1}{n^a \nabla_a t}, \quad N^a(t) = h^{ab} t_b. \tag{3.9}$$

Here $N(t)$ is the *lapse function* which allows you to transport between the spacetime leaves, N^a is the *shift functions* which allows you to move along the surface of the leaf (for a fixed t), t^a are the vector fields defined between the leaves which represents the flow of time through the spacetime $\mathcal{M}$ (with indices a, b), n^a are vector fields which are normal to the leaves, h_{ab} are the metrics induced on the foliation leaves, and ∇_a are the covariant derivatives on the leaves. This decomposition of the spacetime via foliation along with the flow of time through its leaves can be visualized with the following figure:

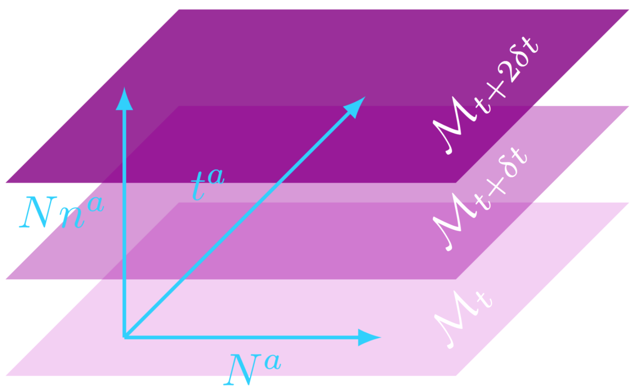

Figure 2. Flow of time through different constant-time hypersurfaces of the spacetime $\mathcal{M}$. Here we can flow between leaves using the lapse function N, move along the leaves with the shift function N^a, and these combined gives a time flow in the spacetime given by t^a. These functions are eliminated by gauge transformations and so do not represent inherent physical DOF; they only give us a prescription for time evolution over leaves of a foliated spacetime.

This reparametrization of time ($t \to N(t)t$) transforms the transition amplitude in the following way:

$$K(\Omega, \mho) = \int \mathcal{D}N(t) \int \mathcal{D}x(t)\ \mathcal{D}y(t)\ e^{iS[N,x,y]}. \tag{3.10}$$

For classical particle evolution we would integrate $N(t)$ over the real line $\mathbb{R}$. This can be deformed to the complex plane $\mathbb{C}$ via analytic continuation to include *classically prohibited* evolutions which can include quantum tunneling events. One could in principle fix $N = 1$ for classical particle evolution, and $N = -i$ for moving in imaginary Euclidean time $\tau = it$ where you can quantum tunnel. When you introduce an imaginary lapse function like $N = -i$ you are essentially modifying the equations governing the evolution of spacetime and particles. This modification introduces complex numbers into the equations which can lead to non-classical, *quantum effects*. For our system the lapse function cycles between classical to quantum to classical evolution: $N(t) \to \{1, -i, 1\}$. For the case of two particles in the lattice wells with a reparametrized theory over hypersurface leaves, we get the action of the system as [29]:

$$S\,[N, x, y; t] = \int dt\ N\left\{\frac{m}{2N^2}(\dot{x}^2 + \dot{y}^2) - V_L(x) - V_R(y) + E_{nk}\right\}. \tag{3.11}$$

On a leaf of the foliation we fix $N(t) = N$. The computation of the path integral in equation 3.10 is very complex but it is sufficient to expand the integrand around its stationary points — as they give the leading contributions in the expansion — which is known as *saddle point expansion*. To be able to do this we need to expand the integrand $\exp(iS)$ in regions where it asymptotically converges and contours in this region go along the direction where the integrand does not strongly oscillate. The question is how do we select contours which go through the saddle points and are in regions of asymptotic convergence $\mathscr{C}$. For this we will make use of *Picard-Lefschetz* theory which selects cycles on contours based on the steepness of the parameter space. We note that we are allowed to *deform* the contour via an extension of the Cauchy integral theorem of it belongs to the same *relative homology* class[†]. Over these cycles of the homology class we select contours where the integrand does not oscillate strongly which are known as steepest descent paths $\Gamma(t)$. In the more general case of a complex integrand functionals, we deal with the generalization of steepest descent paths in $\mathbb{C}$ known as *Lefschetz thimbles*. For all stationary points φ_σ of the complexified action with indices σ, the Lefschetz thimbles $\mathscr{J}_\sigma$ are the unions of all the steepest descent curves that fall in φ_σ for $t \to \infty$ which satisfies [30]:

$$\frac{dz^j}{dt} = -i\frac{\partial \overline{S(z)}}{\partial \bar{z}_j}. \tag{3.12}$$

Here z^j are the complexified trajectories of the action, and $\bar{S}$ is the complex conjugate of the complexified action. In general thimbles are manifolds of real dimension n immersed in $\mathbb{C}^n$. This gives us a prescription on how to select the cycles and contours to integrate the path integral transition amplitude given by equation 3.10: instead of integrating over

[†]Describes the difference between homologies on subspaces of the system.

the full complex plane, we integrate over different the different thimbles which go through the stationary/saddle points φ_σ. This looks like the following:

$$K(\Omega,\mho) = \int \mathcal{D}N \int \mathcal{D}x\, \mathcal{D}y\; e^{iS[N,x,y]} = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \mathcal{D}N \int \mathcal{D}z\, \mathcal{D}\bar{z}\; e^{iS[N,z,\bar{z}]}. \tag{3.13}$$

Here n_σ are integer coefficients which is the intersection number of thimbles $\mathcal{J}_\alpha$ and the steepest ascent paths $\mathcal{K}_\sigma$. This tells us how strongly the real part of the path integrand changes near the saddle points. In such a case we can write the region of asymptotic convergence of the integrand as the sum of the different thimbles $\mathscr{C} = n_\sigma \mathcal{J}^\sigma$. To compute the path integral we expand the integrand functional and compute the classical solutions of the equations of motion (EOM) of the action. To compute the EOM of the action we take the functional variation of the action to vanish $\delta S = 0$. We are left with the following EOM:

$$\begin{gathered}\frac{m}{2}(\dot{x}^2 + \dot{y}^2) + N^2(V_L + V_R) = N^2 E_{nk} \\ \ddot{x} = -\frac{N^2}{m}\frac{\delta V_L}{\delta x}, \quad \ddot{y} = -\frac{N^2}{m}\frac{\delta V_R}{\delta y}.\end{gathered} \tag{3.14}$$

Being that the wells are piece-wise constant, the RHS of the equations of motion for (x, y) vanish. Furthermore for given initial and final positions given respectively by (x_0, y_0) and (x_1, y_1), and the fact that particles swapped position $(y_1 = x_0\ ,\ y_0 = x_1)$, we get the following classical solutions as $x_c(t) = (x_1 - x_0)t + x_0$ and $y_c(t) = (x_0 - x_1)t + x_1$. When plugged in the action in equation 3.11, we get the following lapse function dependent action functional:

$$S[N, x_c, y_c,] \equiv \mathcal{S}(N) = \frac{m}{N}(x_1 - x_0)^2 + N(E_{nk} - V_L - V_R). \tag{3.15}$$

We can compute the saddle points of the functional by solving the equation $\partial\mathcal{S}/\partial N = 0$ which give us the following complex points:

$$\varphi_\sigma = \pm(x_1 - x_0)\sqrt{\frac{m}{E_{nk} - V_L - V_R}}. \tag{3.16}$$

These saddle points have associated thimbles in which $\mathrm{Re}[i\mathcal{S}(N)]$ decreases and increases monotonically on $\mathcal{J}_\alpha$ and $\mathcal{K}_\alpha$. This tells us where to expand the path integrand around — about the saddle points — which have a class of steepest descent curves passing through them (the thimbles). Now we are ready to compute the path integral. Following the approach used in [29], we make use of a *semi-classical* approximation whereby we perturb the classical particle fields with Gaussian fluctuations, giving us: $x(t) = x_c(t) + \mathcal{Q}(t),\ y(t) = y_c(t) + \tilde{\mathcal{Q}}(t)$, where $(\mathcal{Q}, \tilde{\mathcal{Q}})$ are the Gaussian fluctuations about the classical backgrounds (x_c, y_c). Plugging this into the action functional in the path integral, we get the following:

$$K(\Omega,\mho) = \int \mathcal{D}N(t)\exp(i\mathcal{S}(N))\int \mathcal{D}\mathcal{Q}(t)\exp\left(i\int_0^1 dt\,\frac{1}{2}m\dot{\mathcal{Q}}^2\right)\int \mathcal{D}\tilde{\mathcal{Q}}(t)\exp\left(i\int_0^1 dt\,\frac{1}{2}m\dot{\tilde{\mathcal{Q}}}^2\right)$$
$$= \int \frac{dN}{\sqrt{N}}\exp(i\mathcal{S}(N))\left(\sqrt{\frac{m}{2\pi i}}\right)\left(\sqrt{\frac{m}{2\pi i}}\right) = \frac{m}{2\pi i}\sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma}\frac{dN}{\sqrt{N}}\exp(i\mathcal{S}(N)). \tag{3.17}$$

From the first to the second inequality we used a Gaussian path integral identity for the path integrals over the Gaussian fluctuations, and as for the lapse function measure we note that we have fixed $N(t) = N$ and so the path integral measure reduces to the above 1D measure. The third equality comes from reducing the integration domain to be the thimbles associated with the action's stationary points φ_σ. *Now the question becomes: what contours do we select when integrating near the fixed points?* To do so we must look at the regions in the complex lapse-function plane $\Lambda = \text{Re}(N)\times\text{Im}(N)$ where the integrand $\exp(i\mathcal{S}(N))$ weakly oscillates. To see where the steepest path descents are, we plot the real part of the interand which looks like the following[‡]:

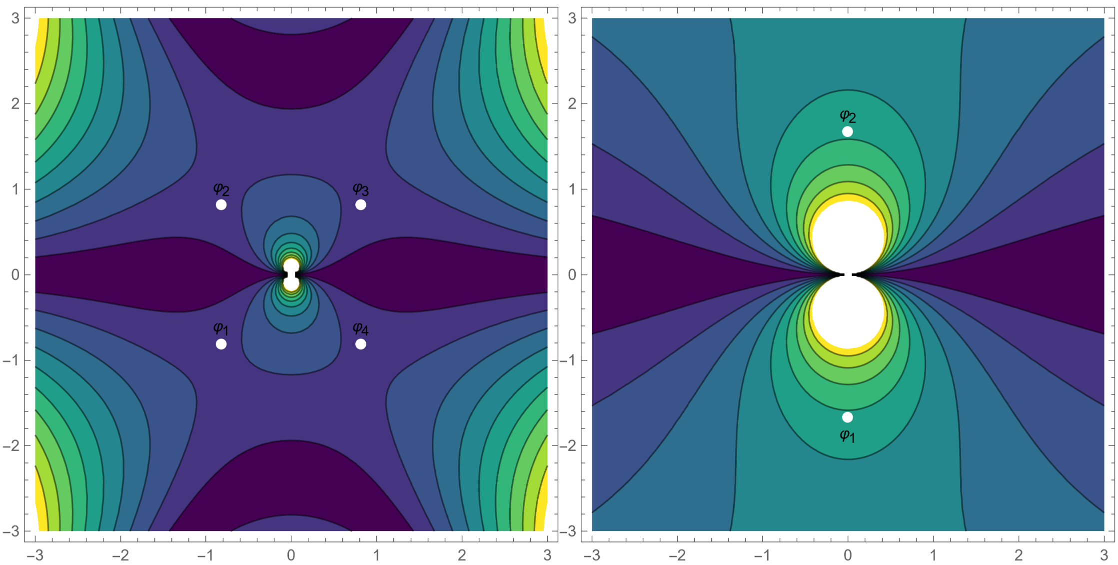

Figure 3. The function $\text{Re}[i\mathcal{S}(N)]$ plotted over the complex lapse-function plane Λ with the corresponding saddle points φ_σ. The left figure is an example plot for a double-oscillator potential while the right figure is for a double-well potential. Depending on the range of N selected, we are allowed to pick a different amount of contours of the thimble.

For the case of the double-oscillator potential, if the range is selected to be $N = (0,\infty)$ then the only contour in the thimble we can select starts in the centre and goes through φ_4 [29]. If the range is instead $N = (-\infty,\infty)$ we can either select a contour that passes through all the saddle points, or one that only passes through the bottom two. The

[‡]To visualize the function over the complex plane, we select the parameters $(m, E_{nk}, x_0, x_1, , V_L, V_R) = (1, 0, 0, 20, 15, 15)$.

different selection of lapse-function ranges and thus contours will contribute differently to the path integral. For our case we want *all* the saddle points to contribute and so we take the full range $N = (-\infty, \infty)$ to allow us to select a contour in the thimble that passes through both saddle points, which also satisfies the condition of steep descent. With this we can expand the action function $\mathcal{S}(N)$ around the saddle points and integrate over the contours. Note that we ignore higher order terms $\mathcal{O}(N-\varphi_\sigma)^3$ as they are much smaller in magnitude in comparison to the leading order terms. To simplify the expression, we can rewrite the polynomial terms of the expansion as $(N-\varphi_\sigma) \equiv Re^{i\theta_\sigma}$ (in essence a variable transformation), which gives us the following result:

$$\begin{aligned} K(\Omega, \mho) &= \frac{m}{2\pi i}\sum_\sigma n_\sigma e^{i\theta_\sigma}\frac{1}{\sqrt{\varphi_\sigma}}\exp\left(i\mathcal{S}(\varphi_\sigma)\right)\int_{\mathcal{J}_\sigma} dR\ \exp\left(-\frac{1}{2}\frac{\partial^2\mathcal{S}}{\partial N^2}\bigg|_{\varphi_\sigma} R^2\right) \\ &= \frac{m}{2\pi i}\sum_\sigma n_\sigma e^{i\theta_\sigma}\left(i\varphi_\sigma\frac{\partial^2\mathcal{S}}{\partial N^2}\bigg|_{\varphi_\sigma}\right)^{-\frac{1}{2}}\exp\left(i\mathcal{S}(\varphi_\sigma)\right). \end{aligned} \tag{3.18}$$

Here to get from the first to the second inequality we computed the Gaussian integral over the measure dR. In our case the intersection number of ascent and descent paths are unity n_σ=1. Furthermore, we compute the derivatives of the action $\mathcal{S}(N)$ and the saddle point phases $\theta_\sigma = \pi/4 - (1/2)\arg(\partial^2\mathcal{S}/\partial N^2|_{\varphi_\sigma})$ which gives us the following result for the particle swapping transition amplitude:

$$\begin{aligned} K(\Omega, \mho) = \frac{m(1+i)}{4\pi i\sqrt{E_{nk}-V_L-V_R}}&\left[\exp\left\{\frac{i\pi}{4}-\frac{ig(1)}{2}\right\}\exp\left\{2i(x_1-x_0)\sqrt{m(E_{nk}-V_L-V_R)}\right\}\right. \\ &\left.+\exp\left\{\frac{i\pi}{4}-\frac{ig(2)}{2}\right\}\exp\left\{-2i(x_1-x_0)\sqrt{m(E_{nk}-V_L-V_R)}\right\}\right]. \end{aligned} \tag{3.19}$$

Here $g(\sigma) \equiv \arg[2m(-1)^\sigma(x_1-x_0)^{-1}(\frac{m}{E_{nk}-V_L-V_R})^{-3/2}]$ is a function defined for convenience. This contributes to the system's partition function non-perturbatively and is in fact an *instanton* contribution. In the following subsection we consider the same setup except where instead of a separable state of two particles, we consider an entangled thermofield double state.

3.2.2 Entangled States and Wormholes

In this section instead of a separable particle state such as $|\Omega\rangle = |n\rangle \otimes |k\rangle^*$ for two particles at a different energy levels, we instead consider an non-separable, *entangled* TFD state of particles at the same energy level. We thus define the initial and final states as:

$$|\Omega\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_n e^{-\beta E_{nn}/2}\,|n\rangle\otimes|n\rangle^*\,,\quad |\mho\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_n e^{-\beta E_{nn}/2}\,|n\rangle^*\otimes|n\rangle\,. \tag{3.20}$$

Here we have the system at a fixed inverse temperature β, and the energy levels simplify due to the fact that both particle states are at the same energy level $E_{nk} \to E_{nn}$. We use a TFD state being that we consider the case where we have doubled the DOF of the system (two copies of the original quantum theory). Unlike the previous section, we can have different non-vanishing values of entanglement entropy which are associated with different entanglement orbits. Why these orbits are of interest is that for a given bipartite system (two particles of some arbitrary level system) we can classify all the possible states of the system efficiently for a given value of entanglement entropy as points on these orbits. The way to flow between different orbits of entanglement is through combinations of local unitary operators [19] and so this gives us a prescription to globally describe the entangled system for different values of entanglement. Moreover these orbits maybe be connected via extremal geometries (such as topological wormholes) due to the nature of having a non-exact symplectic form defined on it and that they are Lagrangian submanifolds of the system's projective Hilbert space. Such a connection would give us additional terms to the partition function and modify the dynamics of the particles but for this section on entangled states in potential-well lattices we will focus on tunneling as the principle non-perturbative correction.

A potential modification would be to the system's Hamiltonian. Usually TFDs are used in the context of the eternal blackhole which has two copies of the same CFT existing on its two conformal boundaries. Being that the time flows in different directions on the two boundaries evolve — and for it to still be dual to a geodesic in a wormhole geometry — we must impose that $H = H_L - H_R$ [23]. This is also convenient as this formalism makes the TFD state time-independent [17]. We don't want this as we want our TFD state to evolve in time and swap after the tunneling event. In our case we have partially separate quantum systems but time flows in the same direction along the different foliation leaves and so we stick with $H = H_L + H_R$. This means that the TFD states we use in this section are *not* dual to geodesics along spacetime wormhole geometries.

Now, being that the system is entangled how do we compute these different entanglement orbits? First, we must compute the entanglement entropy. For this we must consider the Schmidt coefficients (coefficients of the expansion of a general bipartite state) of the entangled state, which are equal to the square root of the eigenvalues of the reduced density operator [31]. This can be obtained by tracing out one of the particle subsystems as follows:

$$\rho^{L/R} = \mathrm{tr}_{R/L}(|\Omega\rangle\langle\Omega|) = e^{-\beta H_{L/R}} \tag{3.21}$$

From this we can compute the Schmidt coefficients κ_n by taking the square root of the eigenvalues of $\rho^{L/R}$ as:

$$\kappa_n^{(L/R)} = \sqrt{e^{-\beta\pi^2k^2/2m\alpha^2}}, \sqrt{e^{-\beta\pi^2k^2/2m(\gamma-\beta)^2}}. \tag{3.22}$$

From this we want to look at the entanglement entropy of the system which tells us how much information is lost when looking at subsystems. We can compute compute the entanglement entropy of the subsystems via:

$$S_L = -\sum_n (\kappa_n^{(R)})^2 \log(\kappa_n^{(R)})^2 = \frac{\beta\pi^2}{2m\alpha^2}\sum_n n^2 \exp\left(-\beta\pi^2 n^2/2m\alpha^2\right)$$
$$S_R = -\sum_n (\kappa_n^{(L)})^2 \log(\kappa_n^{(L)})^2 = \frac{\beta\pi^2}{2m\alpha^2}\sum_n n^2 \exp\left(-\beta\pi^2 n^2/2m(\gamma-\beta)^2\right) \tag{3.23}$$

We can recover the total entanglement entropy stored within both of the subsystems via addition $S = S_L + S_R$. Now, what type of entanglement is our system under? The classification is as follows. If there exists only one non-zero Schmidt coefficient, the state of the entangled system is *separable* and is not entangled. If all the coefficients are identical, then state is *maximally entangled*, and if its somewhere in between then it is simply *intermediately entangled*. The degeneracy of the Schmidt coefficients is labelled by coefficients m_j but for our system without degeneracy the coefficients are unity $m_j = 1$. Furthermore, since our system isn't a finite N-level systems such as in [19, 31] as there is no upper bound on the energy states of the system. Given this, the entanglement orbits reduce to the following quotient space:

$$\mathcal{O} = \lim_{N\to\infty}\left[\frac{U(N)}{\prod_j U(m_j)} \times \frac{U(N)}{U(m_0)\times U(1)}\right] = \lim_{N\to\infty}\left[\frac{U(N)}{[U(1)]^N} \times \frac{U(N)}{[U(1)]^2}\right]. \tag{3.24}$$

Here the first equality has its first element in the base manifold while the second term of the product is in the fibre. Thus locally this gives us a region of the Hilbert bundle with a local trivialization. From this we can obtain the different entanglement orbits for the different entanglement types. For cases of separable, intermediately entangled, and maximally entangled states, respectively the orbits reduce to the elements of the set set:

$$\{\mathcal{O}\} = \lim_{N\to\infty}\left\{CP^{N-1}\times CP^{N-1}, \frac{U(N)}{[U(1)]^N}\times\frac{SU(N)}{\mathbb{Z}_n}, \mathbb{1}\times\frac{SU(N)}{\mathbb{Z}_N}\right\}. \tag{3.25}$$

These different entangled orbits — which are symplectic sub-bundles of the Hilbert bundle — tell us all possible states that could be given for that value of entanglement entropy. What's more is that we can flow between different entanglement orbits via linear combinations of local unitary transformations $U_L \otimes U_R$ which come about when evolving the state in time with an interaction Hamiltonian [19]. Moreover, the fact that the orbits are compact symplectic sub-bundles of the Hilbert bundle, a non-exact symplectic form Ω would elude to extremal geometry of arbitrary topology connecting different independent orbits. This would induce extra terms to the systems partition function path integral and allow the system to evolve different when interacting with each other.

What happens to the transition amplitude with a TFD state? The transition amplitude changes with the replacement of TFD states:

$$
\begin{aligned}
K(\Omega,\mho) &= \frac{1}{Z(\beta)}\sum_{n,k} e^{-\beta(E_{nn}+E_{kk})/2}\,\langle k|^{*}\otimes\langle k|\, e^{-i(H_L+H_R)t}\,|n\rangle\otimes|n\rangle^{*} \\
&= \frac{1}{Z(\beta)}\sum_{n,k} e^{-\beta(E_{nn}+E_{kk})/2}\int_{\Omega}^{\mho}\mathcal{D}x\,\mathcal{D}y\; e^{iS[x,y;E_{nk}]}
\end{aligned}
\tag{3.26}
$$

This sums over the different tunneling amplitudes for the separate particle pairs at different energy levels. This modifies the transition amplitude to:

$$
\begin{aligned}
K(\Omega,\mho) = \frac{m(1+i)}{Z(\beta)}\sum_{n,k}\frac{e^{-\beta(E_{nn}+E_{kk})/2}}{4\pi i\sqrt{E_{nk}-V_L-V_R}}&\Bigg[\exp\left\{\frac{i\pi}{4}-\frac{ig(1)}{2}\right\}\exp\Bigg\{2i(x_1-x_0)\times \\
\times\sqrt{m(E_{nk}-V_L-V_R)}\Bigg\}+\exp\left\{\frac{i\pi}{4}-\frac{ig(2)}{2}\right\}&\exp\Bigg\{-2i(x_1-x_0)\sqrt{m(E_{nk}-V_L-V_R)}\Bigg\}\Bigg].
\end{aligned}
\tag{3.27}
$$

Once again, this gives us a non-perturbative contribution to the partition function path integral as the entangled swap after quantum tunneling through the bulk lattice potential well. The two terms inside inside the argument of the large bracket each contribute as an instanton term, much like in the case of separable particles. In the following section we look at relating the non-perturbative contributions of the topological wormholes and those of quantum tunneling events.

3.3 Wormhole-Tunnelling Correspondence

This section is for the consideration of relating wormholes to quantum tunneling events in a bipartite systems of entangled qubits in potential-well lattices admitting wormholes. The case of a two particle TFD state that swaps position and thus has two particles quantum tunneling, this corresponds to two *instantons* (solutions to the EOM which as critical points of the action that are localized in space and time). Instantons are the leading quantum corrections to the classical behaviour of systems which appear in the path integral. They are used to study tunneling between topological vacua in non-Abelian gauge theory such as Yang-Mills theory where they are self-dual connections in a principal bundle of the system's spacetime. An example of an instanton is the *BPST instanton* that has relevance in string theory which is the classical solution of the Yang-Mills field equations, which is found when extremizing the SU(2) Lagrangian density:

$$
\mathcal{L} = -\frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu}_{a}. \tag{3.28}
$$

Here the instanton is a solution with a finite action as so $F^{a}_{\mu\nu}$ must vanish at the conformal boundary of the spacetime. These instantons act as solutions with non-trivial topology that correspond to gauge transformations at the conformal boundary which cannot be continuously deformed to unity. Being so, we say the BPST solution is topologically stable. *What about its connection to wormholes?* Work has been done looking at spacetime wormhole creation via quantum tunneling [32]. The argument of their work was based on analytically continuing some FRW Lorentzian spacetime to a Euclidean spacetime as $\tau = it$ (to allow for non-classical evolution) and that the geometry of the instanton can develop a neck in the Jordan frame when minimally coupling the QFT to gravity. This process of analytic continuation is exactly the same when we consider the lapse function $N(t)$ being defined over the whole complex plane to allow for classically forbidden particle evolution. These necks which develop during metastable vacuum decay processes can be connected to form the usual wormhole geometry in spacetime. What's more is that the preparation of this instanton geometry is done in imaginary time, and then is evolved with real time, much like is the case for the TFD state which is dual to geodesics in wormholes hypersurfaces. These connected throats represents a double Euclidean instanton which creates a pair of entangled universes [33]. In our case of entangled qubits in potential well lattices, the non-perturbative contributions come from two instantons in the symplectic Hilbert sub-bundles. This arises with quantum tunneling events and so these topological wormholes contributions are a result the two instanton symplectic throats. This demonstrates a relationship between quantum tunneling and topological wormholes which are frameworks that describe separate systems sharing independent information. These topological wormholes that occur in symplectic Hilbert sub-bundles (instead of symplectic phase space sub-bundles) connect different entanglement orbits. Thus we are left with the *topological entanglement structure* of the entangled bipartite system efficiently. All possible states which admit the same entanglement entropy are captured by the entanglement orbits, and separate entangled bipartite quantum systems may interact in the form of topological wormholes. Adding an external interaction to the system allows you to continuously flow to different orbits of entanglement. The addition of topological wormholes add extra terms to the system's partition function due to the fact that these separate bipartite can now non-locally interact. This ultimately modifies that dynamics of the particles in the system non-perturbatively, much like with non-classical evolution through quantum tunneling.

4 Summary

In this report we have reviewed the topological structures wormholes in quantum gravity and their relation to holographic entanglement. We initiated a study on the entanglement structure of qubits in potential-well lattices with topological wormholes which lead to non-perturbative contributions to the systems partition function in the form of topological wormholes and quantum tunneling. In these lattices we demonstrate the relationship between quantum tunneling events and wormholes with the use of instanton contributions in the transition amplitude.

Acknowledgments

I would like to thank Simon Caron-Huot, Mathieu Boisvert, Clément Virally, David leNir, Alexander Kroitor, Hans Hopkins, and Dylan Chaussoy for our discussions related to wormholes in quantum gravity and algebraic topology. I'd also like to thank my supervisor Igor

Boettcher for his keen insights on the framework used from the perspective of condensed matter theory.

A Appendix: Topological Wormhole Structure

The geometric construction of spacetime and topological wormholes relies on distinct frameworks. However instead of delving into both, we can gain insight into the nature of wormholes by focusing solely on topology. By employing concepts such as *foliations* and *cobordisms*, we can develop an intuitive understanding of the mechanisms underlying wormholes and their connection to entangled states.

A.1 Equivalence Classes

First, let's review the concept of *equivalence classes*. Equivalences classes are sets where all the elements are equivalent to each other in some way. Given a set X and some element $a \in X$, the equivalence class of a in X is given by:

$$[a] = \{x \in X : x \sim a\}. \tag{A.1}$$

Here $\sim$ is the *equivalence relation* which tells us how two elements are equivalent (the most common equivalence relation is the equal symbol $=$). Thus, the class $[a]$ is the set of elements of X that are equivalent to a, and is known as a *partition* of X. The set of all equivalence classes or partitions of X is known as the *quotient set*, which is defined as:

$$X/\sim = \{[x] : x \in X\}. \tag{A.2}$$

The quotient set is usually defined between two sets, such as X/Y for some other set Y. This specifies the equivalence relation where two elements of a partition of X are equivalent if they differ by an element of Y. What are some examples of equivalence classes, or more precisely, quotient sets? In the context of topology we can look at *homology* and *cohomology* groups over some manifold M to have some intuition, following [34]. We start with what is known as de Rham cohomology. Let C^p be set of closed p-forms ω_p such that $C^p = \{\omega_p : d\omega_p = 0\}$. Moreover, let E^p be the set of exact p-forms ν_p such that $E^p = \{\nu_p : \nu_p = d\alpha_{p-1}\}$, where α_{p-1} is some $(p-1)$-form. We can construct the de Rham cohomology group as the quotient set:

$$H^p(M) = C^p/E^p. \tag{A.3}$$

Here the elements of H^p are equivalence classes of closed p-forms on M, where the forms are considered equivalent if they differ by an exact form:

$$\omega_p \sim \omega_p + d\alpha_{p-1}. \tag{A.4}$$

The cohomology group actually tells us quite a bit about the topology of M, but this might seem too intuitive to think about this in terms of forms. We thus turn our attention to simplicial homology groups. Consider p-dimensional submanifolds of M, $N_i \subset M$, each labelled by an index i. We can consider a *p-chain* a_p as the sum over the submanifolds of M:

$$a_p = \sum_i c_i N_i, \tag{A.5}$$

where $c_i \in \mathbb{C}$ are coefficients. A *p-cycle* is a p-chain that does not have a boundary such that $\partial a_p = 0$. We can now define the sets which we will quotient to form the homology group. Let C_p be the set of cycles such that $C_p = \{a_p : \partial a_p = 0\}$, and B_p be the set of boundaries such that $B_p = \{b_p : b_p = \partial b_{p+1}\}$. The simplicial homology group of M is the quotient set:

$$H_p(M) = C_p/B_p. \tag{A.6}$$

The elements of the group are equivalence classes of p-cycles of M, where two elements of the equivalence classes are equivalent if they differ by a boundary:

$$a_p \sim a_p + \partial c_{p+1}. \tag{A.7}$$

This can be visualized by the following figure [34]:

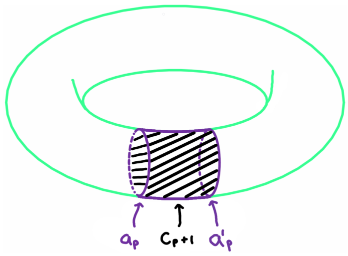

Figure 4. Visualization of the homology of a $(p+1)$-torus. Here a_p and a'_p are p-cycles of the torus, while c_{p+1} is a submanifold of the torus. The cycles a_p and a'_p are equivalent up to the boundary of the submanifold which separates them, given by ∂c_{p+1}. Thus we say a_p is equivalent to $a'_p \equiv a_p + \partial c_{p+1}$ as in equation (A.7).

What does this have to do with topology? Well first off we can construct the topological invariants based on these groups, which are quantities that are preserved under continuous deformations or diffeomorphisms of the space. For example, the dimension of the cohomology groups are the *Betti numbers* given by $b_p = \dim H^p$, which tell us the number of linearly independent harmonic p-forms on M. Additionally, this describes the amount of irreducible p-cycles of M. The connection between the homology and cohomology of M is given by the *Poincaré duality*, which is an isomorphism between the cohomology and homology groups:

$$H^p(M) \cong H_{n-p}(M), \tag{A.8}$$

which holds if M is a compact manifold for $n = \dim M$, and $p \in \mathbb{Z}_+$. Although it might not seem too informative, the cohomology group tells us what forms[†] can exist on M, and the forms correspond to field operators in QFT. These field operators excite the theory's vacuum to give rise to particles, and so we say the topology of the space M tells us exactly what kind of particles can exist on it. An example is the unit 2-sphere S^2 which has the Betti numbers $b_0 = 1, b_1 = 0, b_2 = 1$. Here $b_1 = 0$ tells us that S^2 does not admit a global 1-form or dual vector field, which is a manifestation of the *hairy ball theorem*. This is a direct result of the topology as if we punctured the unit sphere and deformed it to instead be a 2-torus T^2, b_1 would no longer vanish. In essence, the cohomology and homology groups tell us what forms and submanifolds of M are allowed based on its topology.

A.2 Foliations & Cobordisms

Now with some intuition for equivalence classes and topological invariants, we move onto foliations. A *foliation* is an equivalence relation on an n-manifold M through which we decompose M into equivalence classes of its submanifolds. The equivalence classes of submanifolds are known as the *leaves* of the foliation, and we say that M is foliated by the leaves[‡], meaning that we decompose M into its equivalent submanifold constituents. Examples of foliations in physics include foliating a spacetime by decomposing it into constant-time hypersurfaces as seen in figure 5, or the foliation of symplectic spaces when performing geometric quantization.

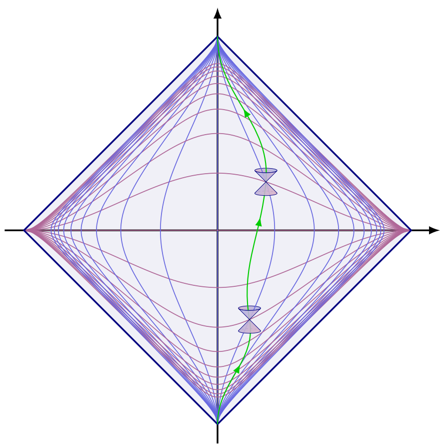

Figure 5. Conformal diagram of the Minkowski spacetime with curvilinear coordinates. The magenta curves represent an equivalence class of constant time slices of the spacetime, while the purple curves represent an equivalence class of constant position slices. The spacetime can be foliated by either equivalence class (the standard convention is foliation with constant time slices). The green curve represents a worldline of an observer with an associated lightcone.

[†]For each p-form we have a corresponding rank p tensor field which is given by the *musical isomorphism* that maps between the cotangent and tangent bundles of M, given by $\sharp : T^*M \to TM$.

[‡]The word is based on the identical structure of tree leaves occurring in nature, given by equivalence classes of leaves for different tree branches. The inclusion of all the branches (partitions) gives the tree leaf structure (quotient group).

Next, we move onto cobordisms which can be used describe wormholes and (entangled) particle creation in topological quantum field theories. A *cobordism* is an equivalence class of compact manifolds of the same dimension. Two manifolds M, N are *cobordant* if their disjoint union is the boundary of a compact manifold W which is one dimension higher; in other words $M \sqcup N = \partial W$. Furthermore, for these manifolds to be cobordant they must share topological properties such as Pontrjagin and Stifel numbers [35]. From this we can construct *cobordism classes* which consist of all the manifolds that are cobordant to a fixed manifold. More explicitly, two cobordisms in this class are considered equivalent if they can be continuously deformed into each other. *How exactly does this tie into the concepts of wormholes?* Consider the spacetime of a maximally extended Schwarzschild solution (the eternal black hole) foliated by constant-time slices. These time slices are extremal hypersurfaces of minimal area that connect regions of space [36], known as wormholes. In this sense, wormholes are cobordisms of spaces with equal dimension such that their disjoint union (the wormhole throat geometry) has minimal surface area. This describes all the different regions of space we could connect that are related under an equivalence. Wormholes belonging to the same cobordism class represent a set of wormholes that can be continuously deformed from one to another [37]. Each wormhole geometry corresponding to a constant-time slice of the spacetime is an element within a cobordism class. This interpretation of wormholes holds for both the spacetime and topological kind.

What about the connection between wormholes and (entangled) particle creation? For this we must extend the framework of topology to include quantum fields by looking at 2+1D topological quantum field theories. In such a theory, observables are topological invariants meaning they do not depend on the spacetime geometry and particles are described by topological defects on compact surfaces [38]. Furthermore, the theory contains topological quantum numbers (topological charges) which are a consequence of the space's topological properties. Consider a compact disk over which we define a quantum field with a globally vanishing topological charge. We then puncture the disk such that it now has an internal and external boundary. Even though we have changed the topology of the disk, the global topological charge still vanishes. We can imagine changing the topology again with another puncture to have two internal boundaries within the disk. Despite the fact that a measurement of the topological charge globally would still result in zero, a measurement on each internal boundary separately could result in a non-zero value, meaning the presence of a topological defect [39] (and hence particles). One boundary can have an associated charge λ while the other boundary has the opposite charge $\bar{\lambda}$ which would preserve the condition of the charge vanishing globally but not locally, meaning we have a topological defect. In this sense this is the creation of a particle-antiparticle pair [39]. Furthermore, if these particles are prepared in such a way where their wavefunction cannot be factorized, they are quantum mechanically entangled. If you plot the spacetime history of these disks (the constant time slices of the spacetime history are the separated disks), we get the following [39]:

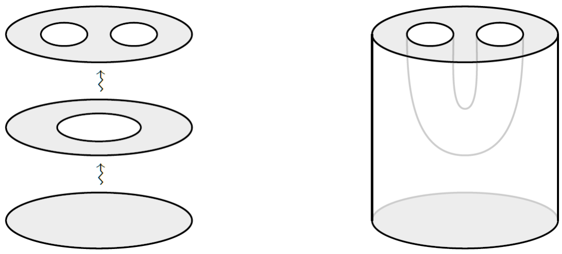

Figure 6. Spacetime history of puncturing a disk. The left plot shows the topological changes of the disk separately, while the right plot describes this change continuously over time as a spacetime history. The resulting configuration for late times corresponds to a pair creation of particles [39].

The spacetime volume can be seen as a cobordism with non-smooth boundary components or corners. The two manifolds which are cobordant in this case are a disk, and a disk with a handle glued to its surface. The cobordism essentially defines an evolution from an initial to final boundary condition of the spacetime. In this sense we could have alternatively taken the disk and created a depression that ends up connecting two pieces of the disk as [39]:

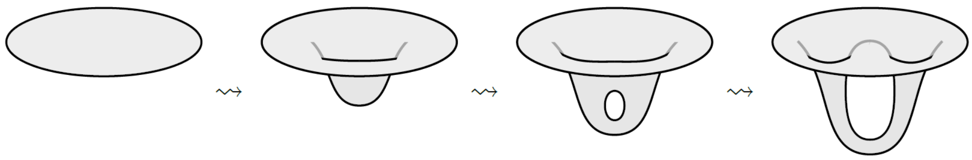

Figure 7. Topological modification of the disk over time. The disk is deformed to have a depression via a diffeomorphism, which is followed by a homeomorphism that allows for a non-trivial genus to form. The resulting geometry connects two regions of the disk via a tunnel [39].

This construction represents a wormhole connecting two regions of the disk. Both the deformations of the disk which gave rise to either particle creation or wormholes result in equivalent spacetime histories. This means that if we plotted figure 7 as a spacetime history, it would result in the identical spacetime volume as in figure 6. This shows us that wormholes correspond to (entangled) particle creation — such as is proposed by the ER=EPR conjecture [5] — via cobordisms.

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