Conflicts with de Sitter vacua in supersymmetric field theories

AUTHOR: **Jeffrey Morais**[a]

SUPERVISOR: **Keshav Dasgupta**[b]

[a] *Physics Student Body, McGill University, Montréal, QC, Canada*
[b] *Department of Physics, McGill University, Montréal, QC, Canada*

Abstract: An enticing goal in physics is to have a consistent model of our universe's geometry through its spacetime manifold. A suitable candidate is de Sitter space dS_4, a vacuum solution of the Einstein field equations, due to its positive curvature and that its an expanding spacetime. Frameworks such as general relativity and quantum field theory are not sufficient to model the universe as it lacks consistency at different energy scales (for example they lack IR/UV mixing). To preserve this consistency, we look at dS_4 as arising from compactifications from type II superstring theory, a supersymmetric field theory with IR/UV mixing. There are however issues with the existence of vacua occurring with de Sitter isometries after compactifying from a higher dimensional theory. These issues prevent non-singular compactifications to dS_4. In particular, we characterize the effect that massive Kaluza-Klein modes have on the theory by compactifying our spacetime over S^1. We then show how effective Wilsonian actions cannot be constructed for expanding spacetimes, implying that a vacuum solution with de Sitter isometries cannot exist. In this paper, we work on constructing quantum corrections to avoid these no-go theorems by replacing these problematic vacua in dS_4 with Glauber-Sudarshan states $|\sigma\rangle$ over supersymmetric Minkowski space $\mathbb{R}^{3,1}$. These states (which are additionally coherent) expand and have the de Sitter isometries, thus are of interest for a model candidate of our universe.

Contents

1 Introduction

It is of great interest in theoretical physics to have a complete theory of our universe which is consistent at different energy scales. It is often the case, however, that high energy theories do not directly affect the features of low energy theories or vice versa; there is no mixing in the degrees of freedom of theories at low energy scales (infrared) and high energy scales (ultraviolet), known as IR/UV mixing. Currently, an example of this energy scale inconsistency which limits us towards the construction of a complete theory is the incompatibility of quantum field theory (QFT) and gravity (as curvature of spacetime in general relativity). Different candidates have been proposed to resolve this conflict of quantum gravity, each running into a variety of inconsistencies. A minimal approach is to construct QFT in expanding curved spacetime but it results in fields having time dependent mass which leads to issues with energy conservation and field quantization. Another minimal approach is to perturbatively add gravity as an interacting term in the theory's action but this leads to a non-renormalizable interaction vertex, meaning observables nonphysically diverge. Another approach is to limit the degrees of freedom of the theory by considering an effective theory. The currently most popular effective theory for quantum gravity is loop quantum gravity but it is constructed in such a way that uses generalized connections (tensor that connects different sub-regions of the spacetime manifolds) which do not reproduce the affine connections used in general relativity. Finally, another approach would be to construct a unified theory which has IR/UV mixing, a popular theory of which is string theory. Although string theory (and hence superstring theory) is self-consistent, meaning alternate methods of computation leads to the same result, we haven't been able to construct a model that reproduces the required properties of nature. On the other hand, many of our theories which *do* reproduce properties of nature, such as QFT or general relativity, reveal many flaws under closer inspection: factorial divergence of loop diagrams or metric singularities from non-vanishing stress tensor components, etc. In this paper we will make use of superstring theory (specifically type II theory in 9+1D) to construct a model of our current universe, which has IR/UV mixing (and hence consistency at different energy scales) and does not run into the same divergence issues with the other theories.

We live in a 3+1D universe/spacetime (3 spatial dimensions $\mathbb{R}^3$ and 1 temporal dimension $\mathbb{R}_+$) where the curvature of the spacetime is very close to being flat. On the other hand, type II string theory is consistent in 9+1 dimensions, i.e. nine spatial and one temporal direction. How then would we get our universe through this higher dimensional theory? The way to recover a lower dimensional theory (such as a 3+1D theory) from a higher dimensional one is by the process of *compactification*: the extra six dimensions form a *compact internal space* which we take to be small. This process, where the internal space is taken to be a compact space, is similar to inverse stereographic projection in mathematics. The physics definition also includes taking the small volume limit of the compact space. In our case for example we could start with type II string theory, a type of string theory which is 10D, and compactify on a 6D (3 complex dimension) complex surface/manifold to recover our 3+1D universe. Here lies the interest of this paper. We want a consistent definition of compactifications from higher dimensional string theories (and their lower energy limit supergravity actions) to field theories on our 3+1D spacetime. A candidate for the space we live in is one which is positively curved and expanding: 4D de Sitter space dS_4. Our goal is to resolve the problems that exist when trying to compactify from string theory to dS_4 such that it can be a candidate for modelling our current universe. In the following we will provide a slightly more technical discussion of our compactification procedure.

As we are dealing with higher dimensional theories, it is often the case that we work with surfaces or manifolds that are embedded in higher dimensional spaces. Such objects are called *hypersurfaces*, and the ambient higher dimensional space (that is equipped with a metric) induces a metric on the embedded hypersurface. The induction of a metric is given by the *pullback* map ϕ_* whereby the metric of the ambient space is pulled back onto the embedded manifold. To outline the construction of a universe model, consider an N-dimensional spacetime manifold Σ_N describing the topology of a universe which is embedded in a higher dimensional universe. The higher dimensional universe has an associated spacetime manifold Σ_M of dimension M. We can write the metric induced in the lower dimensional space as a pullback of the higher dimensional metric as [1]:

$$\phi_*(g_{AB}) = g_{AB}\frac{\partial x^A}{\partial y^\mu}\frac{\partial x^B}{\partial y^\nu} = g_{\mu\nu}, \tag{1}$$

where $(g_{AB}, g_{\mu\nu})$ are the metrics of (Σ_M,Σ_N) each with local coordinates (x^A,y^μ), respectively. The index ranges for the different spaces are $(A, B) = 0, \ldots, M$ and $(\mu, \nu) = 0, \ldots N$, for $N < M$. Being that our current universe is expanding, we are interested in spacetimes with some form of scaling. We can use the previous equation above to write a general form of the expanding metric $g_{\mu\nu}$ on Σ_N, represented in the differential form basis living on fibres of the cotangent bundle $T^*\Sigma_N$:

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = F(t,x) \sum_{j=0}^{N-1} g_{jj} dx^j \otimes dx^j + G(t,x) \sum_{k \neq s}^{N(N-1)} g_{ks} dx^k \otimes dx^s. \tag{2}$$

The first equality shows we are writing the metric in the differential two-form basis $\{dx^\mu \otimes dx^\nu\}$ over the fibres of $T^*\Sigma_N$. The first term in the second equality represents the diagonal terms of the metric, while the second term arises from non-diagonal components. The non-diagonal components have non-trivial mixing of the degrees of freedom (DOFs) of the spatial & temporal components[†]. The sums have been written explicitly contrary to usual convention to emphasize the ranges of the sum. Furthermore, the spacetime functions $F(t,x)$ and $G(t,x)$ scale the spacetime much like the scale factor $a(t)$ does in a Friedmann–Robertson–Walker universe, except now it can have some positional dependence. In models of our universe the metric is diagonal without mixing of DOFs and there is no scaling on the temporal piece of the diagonal components. This simplifies the metric representation to:

$$ds^2 = -dt^2 + F(t,x) \sum_{j=1}^{N-1} g_{jj} dx^j \otimes dx^j. \tag{3}$$

The information of the expansion of spacetime is thus given purely by the scaling function $F(t,x)$, and the nature for the expansion is attributed with the dark or vacuum energy of the theory. In quantum field theory, this vacuum energy is precisely the energy of an unexcited background field, computed via an expectation value with the interacting vacuum state $|\Omega\rangle$ living in the Hilbert space $\mathcal{H}(\Sigma_N)$ (these are section of a Hilbert bundle). The vacuum state energy can be analogously thought as the energy of the ground state of a quantum harmonic oscillator (where in QFT the oscillator modes are instead modes of fields defined over the spacetime manifold). This vacuum energy has a direct correlation with cosmological constant Λ appearing in the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \Lambda g_{\mu\nu} + \kappa T_{\mu\nu}, \quad \Lambda > 0. \tag{4}$$

The above is a non-linear coupled partial differential equation for the metric of the spacetime, whereby one of its vacuum/homogeneous solutions (where $T_{\mu\nu} = 0$) is de Sitter space dS_4. This vacuum solution is not to be confused with the interacting vacuums $|\Omega\rangle$ in QFT which when excited give rise to interacting fields. Explicitly, dS_4 is a *vacuum* solution of the Einstein field equations, where *interacting vacua* $|\Omega\rangle \in \mathcal{H}(dS_4)$ on dS_4 have the isometries of the space, which we call the de Sitter isometries. Even with these isometries, we have yet to construct a consistent description of the vacuum energy relating to Λ as seen in nature. To study the origin of this expansion, and thus the origin of our universe, we look at studying the vacuum energy in various models. Although the universe we live in appears to have a vanishing Riemannian curvature, this is not the case at large scales, and therefore it is informative to study the vacuum energy in a class of expanding, curved geometries whereby fundamental physics might emerge.

For curved geometries such as $AdS_n \times S^n$ where AdS_n is the n-dimensional anti-de Sitter space and S^n is the n-sphere, the cosmological constant is negative $\Lambda < 0$. This occurs in theories of classical supergravity (the study of general relativity over superspaces), however the negative cosmological constant does not lead to an expanding universe. We thus consider geometries admitting $\Lambda > 0$ such as de Sitter space dS_n, which

[†]An example of DOFs mixing is in the Kerr metric that has a component of the form $dt \otimes dx$. This gives rise to non-vanishing angular momentum and as a result the black hole which it models rotates.

give rise to expansion. The geometries with a positive cosmological constant, however, are subject to certain classical conditions [2] whereby the components of the metric on the space after compactification (or the warp factor for warped geometries) diverge. This means that there cannot exist consistent vacuum solutions (de Sitter space) from compactifications of string theory. This leads us to the so-called *swampland* scenario: a model that cannot be realized as a consistent theory of gravity.

Now, if we look at the vacuum energy of supersymmetric Minkowski space $\mathbb{R}^{3,1}$ (a supersymmetric extension of Minkowski space which we use to preserve supersymmetry), because there is no expansion or curvature then the vacuum energy is trivially 0. On the other hand, the vacuum energy of dS_4 must be greater than 0 to lead to $\Lambda > 0$ for an expanding universe, but the interacting vacua $|\Omega\rangle$ are constrained by the lack of classical non-singular compactifications to dS^4. To avoid these issues, we instead work with excited a *Glauber-Sudarshan* (GS) states $|\sigma\rangle$ [3] (until covered in section 2.3 can be thought of general coherent states $|\alpha\rangle$) over supersymmetric Minkowski space $\mathbb{R}^{3,1}$, where the vacuum energy of these computed using these states gives a positive quantity and thus a positive cosmological constant. Although super Minkowski space is flat and does not expand, these GS states are expanding and have the required de Sitter isometries, meaning that the combination of the GS states and Minkowski space are still a relevant candidate to model our universe. It is possible to find these states once quantum corrections to the metric are added to the theory for a consistent model. We wish to characterize the origin of the vacuum energy of these states from compactifications of string theory to dS_4, and hence characterize the origin of the expansion of the universe, via the construction of these quantum corrections. An in depth review of the framework used in the paper appears in the appendix.

2 Analysis of de Sitter vacua

We now present the conflict of interest in more explicit detail to precede the analysis. Recall the interest of this paper, which is to describe quantum corrections to the metric $\langle\sigma|\hat{g}_{\mu\nu}|\sigma\rangle$ (where $\hat{g}_{\mu\nu}$ is the metric operator) that must be added to make a compactification of string theory (type II superstring theory for spectrum of fields we are interested in) to de Sitter space dS_4 consistent. In particular, we want to construct the Glauber-Sudarshan states $|\sigma\rangle$ over supersymmetric Minkowski space $\mathbb{R}^{3,1}$, that have de Sitter isometries and positive vacuum energy. There are multiple ways to consider such conditions on possible compactification solutions.

One approach is to constrict the metric of the different spaces. There exists Hawking-Penrose singularity theorems [4] whereby energy sources in a spacetime will create singularities in the form of diverging metric components, and this cannot be avoided (and thus a singularity is created after compactification with the inclusion of these energy sources). One of the reasons for this inevitability is the requirement of geodesic completeness of the Lorentzian manifold, known as a geodesic manifold. This states that exponential map (the same one used to generate Lie group elements in group theory used in quantum mechanics) must be defined on the entire tangent space of the manifold. An example of a manifold without geodesic completion is the punctured plane $\mathbb{R}^2 \setminus \{0\}$ whereby geodesics going through the origin cannot be defined on all of $\mathbb{R}$. It has been shown that there are conditions on this singularity by which compactifications cannot give rise to dS_4. Such include a *strong criteria* where the g_{00} component of the metric cannot diverge when approaching the spacetime singularity, and a *weak criteria* where g_{00} should be bounded above [2]. For *regular* geometries they find that the sign of the cosmological constant Λ is determined by the sign of the curvature scalar $R(M)$

of the spacetime manifold M, and thus by the contracted Einstein field equations, the trace of the stress tensors T. It is shown that all contributions to the stress tensor are positive and thus contribute to the wrong cosmological constant sign. This follows from a difference of the trace of the stress tensors in the external and internal spaces, respectively, which results in the incorrect negative sign for T. Moreover, for *warped* geometries[†], the only warp factor (a scaling factor of the external space which is dependent on the internal space) solution Ω which is found is one that vanishes near the spacetime singularity: $\Omega \to 0$. This means that at the singularity our external spacetime would vanish and not admit our universe. Thus they conclude there are no non-singular compactifications from string theory to de Sitter space dS_4 [2]. More work has been done in this direction where it was shown that the assumption of the previous work (which was assuming the potential of the scalar field is non-positive $V \leq 0$) does not lead to an external space with positive curvature such as dS_4 [3]. Moreover, they further this analysis by including fluxes coupled to gravity, (anti) Dp-branes and (anti) orientifold planes Op. It is found that the inclusion of such objects, even in warped geometries, does not give rise to a positively curved universe and quantum corrections are required in the form of higher derivative gravity terms as well as non-perturbative corrections in string theory.

An alternative approach would be to have a metric independent argument using different spacetime energy conditions. The energy conditions that exist for spacetimes include: the *null energy condition* where parallel lightrays in a curved spacetime cannot diverge, the *strong energy condition* where gravity is non-repulsive, the *weak energy condition* where a local observer must see a non-negative energy density and non-repulsive gravity, and finally the *dominant energy condition* where a local observer sees a causal flow of non-negative energy and non-repulsive gravity [5]. Work has been done to ascertain whether these energy (consistency) conditions in a higher dimensional theory imply the inheritance of the same energy conditions in a lower dimensional theory after compactification. It is shown that the inclusion of fields (and hence fluxes), branes, and orientifold planes together satisfy the energy conditions, and that the higher dimensional energy conditions indeed imply the inheritance of those conditions in the lower dimensional theory [5]. This work however did not include perturbative and non-perturbative corrections arising from string theory which can change the consistency of these energy conditions. This changes the criteria in which compactifications to dS_4 are allowed.

In either case we wish to construct the conditions by which we can have an expanding geometry with positive curvature after compactifications of string theory, with the inclusion of missing quantum corrections on the metric. This would allow us to construct the Glauber-Sudarshan states and see the conditions on when the sign of the cosmological constant is positive. First we make sure that the massive spectrum of states that occur after compactification have vanishing contributions to the metric. Should they be non-vanishing, the states would curve the spacetime which alters the metric and thus does not give us a de Sitter spacetime. Then we show that in expanding spacetimes, we cannot define an effective Wilsonian action (meaning we cannot define a vacuum in the theory). Then we look at quantum corrections to circumvent the constricting conditions of dS_4 compactifications by considering the interacting vacua as excited Glauber-Sudarshan states $|\sigma\rangle$ over a flat supersymmetric Minkowski space background $\mathbb{R}^{3,1}$.

[†]In warped geometries where the metric is split into a piece for the internal space, and another piece for the external space (our universe), the external piece has a scaling factor which depends on the coordinates of the internal space. This is known as the warp factor for a warped product metric.

2.1 Kaluza-Klein state spectrum

Although the fields we have seen thus far are massless and Abelian, there does exist a massive sector of the theory consisting of different sets of massive fields. These are relevant to our interests as the presence of massive objects in our spacetime can alter the metric and alter the construction of the de Sitter spacetime dS_4. One such set of massive fields arise from compactification which are known as Kaluza-Klein states, which we must study if we want to recover a consistent theory for our 3+1 D spacetime. We must make sure these massive DOFs don't change the physics of our theory.

To see if this is the case, consider an ambient 4+1 dimensional spacetime of form $\Sigma' = \Sigma_4 \times S^1 = (\mathbb{R}^3 \times \mathbb{R}_+) \times S^1$, where the first term Σ_4 is a 4 dimensional Lorentzian manifold defining the 3+1 dimensional spacetime we live in, and S^1 is the unit circle which we will compactify over. We define tensor field components over Σ' to be $(M, N) = 0, \ldots, 4$ with coordinates $z = (x^0, \ldots, x^4)$. We can additionally define components for each separate space as $(\mu, \nu) = 0, \ldots, 3$ for Σ_4 with coordinates $x = (x^0, \ldots, x^3)$ and $i = 4$ for S^1 with coordinates $y = x^4$. The presence of scalar and gauge fields defined over S^1 adds an extra term in the metric as the following:

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu + e^{2\varphi}(dy + A_\mu dx^\mu)^2 \tag{5}$$

Here $g_{\mu\nu}$ is the metric over the external space, dx^μ is the basis for cotangent spaces $T^*_p\Sigma_4 \ \ \forall p \in \Sigma_4$ (fibres of cotangent bundle $T^*\Sigma_4$), φ is a scalar field defined over Σ', and A_μ is the gauge field defined over Σ_4. We want to study the dynamics of a scalar field over the product space, so we define it as the tensor product of the fields $\varphi_1(x) \in \Sigma_4$ and $\varphi_2(y) \in S^1$ as:

$$\varphi(x, y) = \varphi_1(x) \otimes \varphi_2(y). \tag{6}$$

Although the above are scalars and thus the tensor product is simply multiplication, we keep the notation to emphasize that the fields live in different spaces. Recall that in QFT we write the scalar field as a classical solitonic background $\varphi_0 \in \mathbb{C}$, and we quantize fluctuations $\delta\varphi$ on that background to give us $\varphi = \varphi_0 + \delta\varphi$. It is standard to make the background field vanish and only look at the fluctuations, however we will only look at the background terms, the reason for which will be evident soon. For a φ^4 interacting scalar field theory, we have the usual action and the associated equations of motion of φ:

$$\begin{aligned} S &= \int d^4x \, dy \left[\partial^M\varphi\partial_M\varphi - m^2\varphi^2 - \lambda\varphi^4\right] \\ 0 &= \frac{\delta S}{\delta\varphi} = (\Box + m^2)\varphi + \lambda\varphi^3, \end{aligned} \tag{7}$$

where $\Box = g^{MN}\partial_M\partial_N$ is the d'Alembertian operator, $\lambda \ll 1$ is a dimensionless coupling constant, and the potential $\lambda\varphi^4$ is a perturbative interaction term which maybe be dropped in the EOM. This gives us the usual Klein-Gordon equation (KG) which describes the dynamics of the scalar field in the ambient space which it lives in. Now, due to the nature of our space (a product space), the metric will take a different form and will affect the differential operator in the EOM. Analogous to how we decompose the metric over a product space in the appendix, we can construct a metric which has a classical background component and a fluctuating component:

$$
\begin{aligned}
g_{MN} &= g^{(b)}_{MN} + \delta g^{(f)}_{MN} \\
&= g^{(b)}_{\mu\nu}(x) \oplus g^{(b)}_{ii}(y) \oplus g^{(b)}_{\mu i}(x,y) + g_{\mu\nu}(x) \otimes \varphi_2(y) \oplus A_\mu(x) \otimes A_i(y) \oplus \varphi_1(x) \otimes g_{ii}(y).
\end{aligned} \tag{8}
$$

Here g_{ab} are metrics with legs (indices) in different parts of the product space, $\varphi_a(x^a)$ are scalar fields, and A_a are gauge fields, for some arbitrary indices (a,b). The first 3 terms are the classical background pieces, while the second set of more complex terms are the fluctuating pieces (where the two objects in each tensor products have the appropriate amount of indices to match up with the metric g_{MN}). The last term in the background pieces $g_{\mu i}(x,y)$ is a metric term with DOFs mixing from both spaces, however as we are not dealing with a fibration but a product space, this term vanishes. Moreover, we shall drop the fluctuating terms as the presence fields within it would turn the EOM into a coupled set of non-linear PDEs, and it suffices to look at the behaviour of the field over a background to see the massive modes don't affect the metric. With this we can decompose the d'Alembertian operator defined over the two spaces:

$$
\Box = g^{MN}\partial_M\partial_N = g^{\mu\nu}\partial_\mu\partial_\nu \otimes \mathbb{1} \oplus \mathbb{1} \otimes g^{ii}\partial_i\partial_i = \Box_x \otimes \mathbb{1} + \mathbb{1} \otimes \Box_y. \tag{9}
$$

The $\mathbb{1}$ are the identity operators in that piece of the product space (either in the internal or external space), and ∂_a are the usual partial derivatives which form a basis for fibres of the tangent bundle $T\Sigma'$. Although S^1 has no temporal coordinate, we write $\nabla^2_y = \nabla^2_y - 0(\partial^2_t) \equiv \Box_y$ for convenience. With the metric and differential operator appropriately decomposed, we can use this to compute the dynamics of the scalar field through its EOM:

$$
\begin{aligned}
0 =& (\Box + m^2)\varphi = (\Box_x \otimes \mathbb{1} + \mathbb{1} \otimes \Box_y + m^2)\varphi_1(x) \otimes \varphi_2(y) \\
&= \Box_x\varphi_1(x) \otimes \varphi_2(y) + \varphi_1(x) \otimes \Box_y\varphi_2(y) + m^2\varphi_1(x) \otimes \varphi_2(y).
\end{aligned} \tag{10}
$$

With a few steps of algebra we can get the equation of motion in the following form:

$$
\Box_x\varphi_1(x) \otimes \mathbb{1} + \varphi_1(x) \otimes \left[m^2 + \frac{\Box_y\varphi_2(y)}{\varphi_2(y)} \right] = 0. \tag{11}
$$

Since the scalar field φ_1 in $\varphi = \varphi_1 \otimes \varphi_2$ resides in the massless sector of type II string theory (type II supergravity which we elaborate in the appendix), we take $m^2 \to 0$, where m is the mass of the scalar field φ_1. Now, although we take the scalar field φ_1 to be massless, the EOM above with $m = 0$ looks like the KG equation for φ_1 with a mass term given by $\Box_y\varphi_2(y)/\varphi_2(y)$. We define this mass term for φ_1 as $\bar{m}^2$, and equating the mass term to its definition gives us a harmonic PDE for φ_2:

$$
\Box_y\varphi_2(y) = \bar{m}^2\varphi_2(y). \tag{12}
$$

This tells us that S^1 admits a *harmonic* 0-form φ_2, which is normalizable as the space is compact. Using the periodic boundary condition on the harmonic scalar field given by $\varphi_2(y + 2\pi R) = \varphi_2(y)$, where R is the radius of S^1, we can solve the differential equation to get:

$$
\varphi_2(y) = c_1 e^{iky} + c_2 e^{-iky}, \tag{13}
$$

where $c_j \in \mathbb{C}$ are complex coefficients, $k \equiv \bar{m} = n/R$ is the mass of the *Kaluza-Klein states* $\varphi_2(y)$, and $n \in \mathbb{Z}$. This harmonic scalar field gives us an infinite tower of massive modes with mass $\bar{m}$ which depend on the parameter n. In principle this would affect the energy stress tensor $T_{\mu\nu}$ in the Einstein field equations and thus affect the metric of dS_4. In our case, does it actually affect the metric? Consider the 2-point function $G^{(2)}(p)$ with total incoming 4-momenta p, for the interaction of two scalar fields via a φ^4 interaction. Looking at the tree level diagrams, the virtual particle mediating the interaction will have a propagator for $\varphi_2(y)$ of the form $D(p;\bar{m}) \propto 1/(p^2+\bar{m}^2) \simeq 1/\bar{m}^2 = R^2/n^2$ (where $\bar{m}^2 \gg p^2$). This term becomes very small as $R \to 0$ as we compactify over S^1 and thus the contribution vanishes. This also follows for loop diagrams with the same contribution vanishing under the compactification. Thus we see that the infinite tower of modes generated by the harmonic scalar field $\varphi_2(y)$ over the compactified manifold S^1 don't get produced and don't affect the geometry of the space at low energies.

An alternate way to get the above is looking at the action under the presence of Fourier modes living in S^1. Due to the identification of points on S^1 via the equivalence relation $S^1 : y \sim y + 2\pi R$ and thus $\varphi(y) \sim \varphi(y+2\pi R)$, we can decompose the scalar fields in the following way:

$$\varphi(x,y) = \sum_{n=-\infty}^{\infty} \varphi_n(x) e^{iny/R}, \tag{14}$$

where the φ_n are the eigenstate component weights of the expansion. Plugging this in the usual action for a massless scalar field over the two spaces results in:

$$\begin{aligned} S &= \int d^4x dy\, \partial_M \varphi \partial^M \varphi = \int d^4x dy \left[\partial_\mu \varphi \partial^\mu \varphi + (\partial_i \varphi)^2\right] \\ &= 2\pi R \sum_{n=-\infty}^{\infty} \int d^4x \left[\partial_\mu \varphi_n \partial^\mu \varphi_n + \frac{n^2}{R^2}|\varphi|^2\right], \end{aligned} \tag{15}$$

and thus we see the contribution from the scalar fields living on S^1 gives rise to a mass term with $m^2 = n^2/R^2$. This mass term of course vanishes under the compactification of S^1.

Now, we have seen that a compact space (in our case S^1) admits a harmonic 0-form $\varphi_2(y)$. It turns out to be true for the more general case where compact spaces always admit normalizable harmonic forms $\{\Omega^{(n)}_{\mu\ldots\nu}\}$, where n is the tensor rank of the form. These harmonic forms satisfy $(dd^\dagger + d^\dagger d)\Omega^{(n)} = 0$ (where $d^\dagger = (-1)^{n(4-n+1)} \star d \star$ and $\star$ is the Hodge star operator), and are normalized according to their scalar product $\langle \Omega, \Omega \rangle \propto \int \Omega \wedge \star\Omega$, which is integrated over the volume of the internal space. Being that the space is compact, the normalization is finite. The requirement of the internal space to be compact, among many other reasons, is to keep the gravitational Newton constant G finite. These harmonic forms in the internal space are exactly the ones that are attached to the forms in the external space via tensor products. As an example, consider the decomposition of a rank-3 (tensor) form over a product space $\mathbb{R}^{3,1} \times \Sigma_6$ (where Σ_6 is compact):

$$C_{MNP}(x,y) = \varphi \otimes \Omega^{(3)}_{mnp} \;\oplus\; A_\mu \otimes \Omega^{(2)}_{mn} \;\oplus\; B_{\mu\nu} \otimes \Omega^{(1)}_{m} \;\oplus\; C_{\mu\nu\rho} \otimes \Omega^{(0)}. \tag{16}$$

Here $\{\phi, A_\mu, B_{\mu\nu}, C_{\mu\nu\rho}\}$ are the forms living in the external space $\mathbb{R}^{3,1}$, while $\{\Omega^{(n)}_{\mu\ldots\nu}\}$ are the harmonic forms living on the compact internal space Σ_6. This exercise also tells us how many forms (or their field duals) can be defined over the manifold in question. For example, we can look at the topology of the manifold by looking at the Betti numbers $b_p = \dim H^p$, where b_p are the dimensions of the the cohomology groups

H^p on the manifold Σ'.[†]

For S^1 we have the following numbers:

$$b_0 = 1,\ b_1 = 1,\ b_{2\leq j\in\mathbb{Z}} = 0, \tag{17}$$

where b_0 tells us the amount of 0-cycles (and thus 0-forms)[‡] over S^1 which in our case is the single harmonic form $\varphi_2(y)$, while b_1 tells us how many 1-cycles (and thus 1-forms) are defined on S^1. From knowing the amount of harmonic forms defined on the manifold, we can count how many fields should be defined on it as well. Consider an antisymmetric 2 tensor defined over the product space Σ':

$$B_{MN}(x,y) = B_{\mu\nu} \otimes \varphi_2(y) \oplus \ldots. \tag{18}$$

Here we can see how many antisymmetric rank-2 tensors $B_{\mu\nu}$ are on the surface by looking how many 0-forms $\varphi_2(y)$ we have, which in the case for S^1 by $b_0 = 1$, is 1.

2.2 Incompatibility of effective Wilsonian actions

In this section we show that we cannot write down an effective Wilsonian action in an accelerating spacetime background due the temporal dependence of the frequencies. This means we cannot integrate out higher energy modes of fields and thus cannot define a theory at a fixed energy scale. This shows that vacuum solutions of the Einstein field equation (de Sitter space being one of them) cannot consistently occur when the solutions are expanding spacetimes. We review this for bosonic fields by following the work done in [7], and then present our results for fermionic fields in an expanding 4-dimensional spacetime manifold.

2.2.1 Scalar fields

Quantizing a bosonic scalar field ϕ on some static flat geometry, such as Minkowski space $\mathbb{R}^{3,1}$, we get the usual Fourier expansion for the quantized scalar field in terms of the creation and annihilation operators $(a_k^\dagger, a_k)$:

$$\phi(t,x) = \int \tilde{dk} \left(a_k e^{ikx} + a_k^\dagger e^{-ikx} \right), \tag{19}$$

where $\tilde{dk} \equiv d^3k/((2\pi)^3\sqrt{2\omega_k})$ is the Lorentz invariant measure, $\omega_k^2 = k^2 + m^2$ is the frequency of the excitations, and $kx = k^\alpha x_\alpha = \omega t - \vec{k}\cdot\vec{x}$ is the sum over the components of the momentum k^α and position x^α of the field excitation.

For an expanding, curved geometry, we first switch to *conformal* time $\eta = \eta(t)$ such that $d\eta(t) = dt/a(t)$, which takes into account the expansion of the universe through the scaling factor $a(\eta)$. We use this to define an auxiliary scalar field $\varphi \equiv a(\eta)\phi$, which will be our description of scalar fields on the expanding curved geometry. It turns out that these auxiliary fields defined on this spacetime, given a time η, have a similar Fourier mode expansion [7]:

[†]This tells us how many linearly independent harmonic p-forms exist over Σ', as well as the number of boundaries of p-dimensional submanifolds M_p, given by $\partial M_p \in \Sigma'$, which themselves do not have a boundary. These boundaries are known as *p-cycles*[6].

[‡]We can relate the amount of n-cycles over the manifold and the number of n-forms via the homology groups H_i. See Candelas' lectures on complex manifolds for an in depth review.

$$\varphi(\eta, x) = \int \overline{dk} \left(a_k v_k^*(\eta) e^{ikx} + a_k^\dagger v_k(\eta) e^{-ikx} \right). \tag{20}$$

Here $\overline{dk} \equiv d^3k/((2\pi)^{3/2}\sqrt{2})$ is another normalized Lorentz invariant measure, and $v_k(\eta)$ are Fourier mode functions satisfying:

$$v_k'' + \omega_k^2 v_k = 0, \tag{21}$$

for $\omega_k^2 = k^2 + m_{\text{eff}}^2 = k^2 + m^2a^2 - a''/a$. The effective mass term in the frequency comes from the scalar field living on an expanding spacetime, which can be imaginary for fields interacting with the gravitational background. Furthermore, it is often the case that we use a different coordinate system that maximizes the amount of variables that are cyclic, (and for each cyclic coordinate comes a conserved charge) while preserving the forms of the equations of motion. In our case, we change Hilbert space bases generated by the creation and annihilation operators which is done through a canonical transformation on the creation and annihilation operators, known as the *Bogoliubov* transformations [8]. This transformation defines an isomorphism between canonical (anti)commutation relation algebra. When performing such a transformation gives us yet another solution to the Fourier mode (Eq. 20) as:

$$u_k(\eta) = \alpha_k v_k(\eta) + \beta_k^* v_k^*(\eta). \tag{22}$$

Here (α_k, β_k) are complex coefficients that depend on the Wronskian of (u_k, v_k), which transform our creation and annihilation operators as:

$$b_k^\dagger = \alpha_k^* a_k^\dagger - \beta_k^* a_{-k}, \quad b_k = \alpha_k a_k - \beta_k a_{-k}^\dagger. \tag{23}$$

Using these operators $(a_k^\dagger, b_k^\dagger)$, we can generate a basis for the Hilbert space of our spacetime. Note that the relationship between the particles created by $a_k^\dagger$ and the particles created by $b_k^\dagger$, is that their respective vacua contains the other particles. So the vacuum with respect to b_k contains particles particles created by $a_k^\dagger$, and vise versa. Where in lies the problem with such vacua? The effective mass term in the frequencies of excitation m_{eff} depends on time, and thus the annihilation operators which annihilate the vacua at one time η_0 will not annihilate it at another η_1. This means we cannot have a consistent description of the vacuum. Moreover, the time dependence of the frequencies ω_k creates an ambiguity in what modes are considered high and low energy. We might try to integrate out the high energy modes via Wilsonian integration to have an effective action of the theory, but the kept low energy modes can become high energy modes at some later time. This means we cannot describe the field dynamics via an effective action, nor can we define a vacuum state which will allow us to compute the vacuum energy (again, related to the cosmological constant Λ).

Now, we see that we cannot define scalar field theory on an expanding curved spacetime, meaning we cannot define a consistent vacuum with the desired de Sitter isometries. It is noted that a popular candidate for a vacuum with these isometries is the *Bunch-Davies vacuum*, which allows us to characterize the behaviour of φ around the event horizon of de Sitter space. Although containing the required isometries, this vacuum assumes that there was a period of exponential expansion during inflation whereby quantum fluctuations are taken to be negligible, which isn't necessarily the case. This, among the other problems associated with de Sitter compactifications mentioned before, leads us to instead consider product coherent states $|\sigma\rangle$, known as Glauber-Sudarshan states, over a flat supersymmetric Minkowski background $\mathbb{R}^{3,1}$. This is covered in the next subsection, after moving on to the analogous problems with spinors on a expanding curved spacetime.

2.2.2 Spinor fields

We now move onto looking at how fermionic spinor fields behave over a curved, expanding spacetime. We see that it doesn't allow us to define a vacuum (and hence the notion of a particle) in the theory or define an effective Wilsonian action, where we have integrated out the high energy fields that cause observables to diverge.

First, we must consider the space we are working with; one which is curved and expands to model our universe spacetime. For curved spaces with complicated geometries, it is useful to work with *fibre bundles* that generalize the notion of produce spaces between two manifolds. In the case of a bundle, the definition of the product between elements of the two manifolds can change at different points and these bundles allows us to define more general fields (of equal footing) in unified theories. The set of two manifolds is composed of a *base* space and a *projected* space, where the projected space space tells you how to select information of the base space. A useful property of fibre bundles is that locally the space looks like a usual product space which allows us to work in a locally flat spacetime (through defining a local trivialization, making use of locally flat charts of $\mathbb{R}^n$). This non-trivial product of two manifolds represents a *fibration*, a more general fibre bundle that doesn't require compact and smooth manifolds.† These manifolds are related by a projective mapping π which acts on the elements of the base space, known as *fibres*. One for example could consider a *spacetime* bundle, where the base space is $\mathbb{R}^3$ and the projected space is $\mathbb{R}^1$ which represents a temporal dimension. Furthermore, different fibre bundles are *associated* with one another if we can define a morphism between them such that we can construct one bundle from the other. One bundle we can have associated to the spacetime bundle is a *vector* bundle where the fibres are vector spaces, over which fields (and hence particles) are defined on. In our case of type II superstring theory that is defined in 9+1D, then we have the spacetime split up as a fibration where the base (external) space is our 3+1 Lorentzian spacetime L^4, and the other (internal) space on which L^4 is fibred over is a 6D compact space M^6 over which we compactify over to get the de Sitter vacua (and hence de Sitter space).

Now, there are a few conditions which must be met to study spinors (elements of complex vector spaces which describe particles spin) on curved, expanding spacetimes. When we looked at the work on scalar fields done in [7], we had a vector bundle associated with our spacetime bundle over which we can define the fields in our theory. Now, as we want to study spinors over our spacetime, we require an associated complex vector bundle, known as a *spinor* bundle. Here we define spinors using the spin representation of the Lorentz group as *sections* of the spinor bundle. Sections are the union of different fibre pieces at different projected space points. Being that it is a curved space, we require a connection under which we can study how the derivative of fields transform at different parts of the space. This is given by the covariant derivative, that acts on these spinors under the representation:

$$D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}. \tag{24}$$

Here ∂_μ is the usual spacetime derivative, ω_μ^{ab} is the *spin connection*, $\gamma_{ab} = \gamma^{[a}\gamma^{b]}$ is the anti-symmetrization of γ matrices, and together this tells us how the spinor field changes on the space. Furthermore, much like

†One for example could think about the product space between the unit circle S^1 and the line $\mathbb{R}^1$, which traces out a cylinder $S^1 \times \mathbb{R}^1$. We can have instead have the fibration $S^1 \rtimes \mathbb{R}^1$ such that at each point on $\mathbb{R}^1$, S^1 shrinks and at a certain point has trivial size. Then after that point it can increase to grow back to the original size. This would look like a cylinder where the center is warped to have zero size in the middle along $\mathbb{R}^1$.

how it is useful to work in a locally flat patch of a curved manifold, it is useful to work on a locally flat inertial frame of the curved spacetime. This is given by the *frame* bundle, a principal bundle (a fibre bundle that has a group action on a fibre space) that is associated with our spinor bundle. The use of the frame bundle is that it allows us to attach a local frame / coordinate basis to each fibre in the spinor bundle. This allows us to describe the space locally with flat, expanding coordinates given by *tetrad* coordinates e^{μ}_{a} which are sections of the frame bundle that have a Lorentz index μ and a basis index a. Tetrads are useful because they allow us to write our metric as locally related to the flat Minkowksi metric, bypassing the need for local trivializations (to work in flat space). This is given by the pullback map $\phi_*(g_{\mu\nu}) = e^{\nu}_{b} e^{\mu}_{a} g_{\mu\nu} = \eta_{ab}$. We can also write an explicit form for the spin connection using tetrads as: $\omega^{ab}_{\mu} = 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_{\nu}e^{c}_{\sigma}$ [9]. One particular use of tetrads for spinors is that we can define a notion of γ matrices in curved spacetime as:

$$\Gamma^{\mu}(x) = e^{\mu}_{a}(x)\gamma^{a}, \tag{25}$$

where γ^{a} are the usually spatially constant γ matrices, and e^{μ}_{a} are the local tetrad coordinates. Together this gives us a notion of spatially dependent gamma matrices. We also note that the form of the covariant derivative with the spin connection comes from the fact that the tetrad coordinates must be *covariantly constant*: $D_{\mu}e^{a}_{\nu} = 0$ [9]. The dynamics of the spinors ψ in a curved, expanding spacetime, together with the gamma matrices and covariant derivatives, is thus given by the following action:

$$S = \int d^4x\sqrt{-g}\left[\bar{\psi}\Gamma^{\mu}D_{\mu}\psi - m\bar{\psi}\psi\right]. \tag{26}$$

Here $\sqrt{-g}$ is the root determinant of the metric (where the minus sign keeps the determinant real) that keeps the measure Lorentz invariant, ψ are the spinor fields, $\bar{\psi} = \psi^{\dagger}\gamma^{0}$ are conjugate spinor fields, Γ^{μ} are the γ matrices in curved spacetime, D_{μ} is the covariant derivative associated with the spin connection ω^{ab}_{μ}, and m is the mass of the spinor fields. Much like how there cannot be a vacuum defined for scalar fields on a flat, expanding spacetime, it is sufficient to show that we can't describe spinors on a curved, expanding spacetime by studying how it fails in a flat, expanding spacetime. This leads us to not be able to have an effective Wilsonian action and the description of a vacuum state on curved, expanding spacetimes. To see how the dynamics of these spinor fields change, we need to see how the expanding metric contributes to the action on this flat space. Consider a metric for an expanding flat universe, given by an FRW metric:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i \otimes dx^j. \tag{27}$$

Here $a(t)$ is the scale factor that determines how the universe expands, δ_{ij} is the flat metric for the spatial component of the space, and $\{dx^i\}$ is the differential form basis of the cotangent space (forming the cotangent bundle of the spacetime). Although we make use of tetrads for describing γ matrices on curved spacetime, we don't need to make use of the pullback map to have a local description of Minkowski space. This is because the FRW metric is already conformally related to the Minkowski metric under the replacement of time t to conformal time η (through $dt = a(\eta)d\eta$), which gives us the metric:

$$ds^2 = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i \otimes dx^j). \tag{28}$$

This allows us to use the replacement $g^{\mu\nu} = a^{-2}\eta^{\mu\nu}$. From the above we can compute the root determinant contribution $\sqrt{-g}$ to the action (where $g = \det(g_{\mu\nu})$) as $\sqrt{-g} = a^4$. Moreover, we can expand out the index contraction of the covariant derivatives as:

$$\Gamma^\mu D_\mu = a^{-2}\eta^{\mu\nu}\Gamma_\mu D_\nu = a^{-2}(-\Gamma_0 D_0 + \delta^{ij}\Gamma_i D_j) = a^{-2}(\Gamma^i D_i - \Gamma_0 D_0). \tag{29}$$

Since the space is flat the spin connection ω_μ^{ab} vanishes and so the covariant derivatives D_μ reduce to normal spacetime derivatives ∂_μ. This means that the above covariant derivatives act on the spinor fields as $D_0\psi = \partial_0\psi \equiv \dot{\psi}$ and $D_i\psi = \partial_i\psi \equiv \psi_i'$. Here ψ_i' is written to mimick usual notation for spatial derivatives such as $f'(x) = \partial_x f$ and is not to be confused with the ψ^ν vector-spinors (also known as *gravitinos* since their are the superpartner to gravitons). With all of the above the action becomes (although we are in flat space, we keep the vanishing spin connection terms to emphasize the structure of the action when we split spatial and temporal indices):

$$S = \int d^4x a^2 \left[\bar{\psi}\left(\Gamma^i\psi_i' - \Gamma_0\dot{\psi}\right) + \frac{1}{4}\bar{\psi}\left(\Gamma^i\omega_i^{ab} - \Gamma_0\omega_0^{ab}\right)\gamma_{ab}\psi - ma^2\bar{\psi}\psi\right], \tag{30}$$

where we have the Dirac conjugate for curved spacetime: $\bar{\psi} = \psi^\dagger\Gamma^0$. Now, we introduce the notion of an auxiliary field $\chi = a(\eta)\psi$ that will be our description of a spinor field on the expanding spacetime. Taking spatial and temporal derivatives of χ and relating it to the original spinor field gives us the following relations:

$$\dot{\psi} = \frac{\dot{\chi}}{a} - \frac{\dot{a}}{a^2}\chi, \quad \psi_i' = \frac{\chi_i'}{a} - \frac{a_i'}{a^2}\chi. \tag{31}$$

Plugging the expressions for $(\psi, \dot{\psi}, \psi_i')$ in the action and after some algebraic manipulations we are left with:

$$S = \int d^4x \left[\bar{\chi}\left(\Gamma^i\chi_i' - \Gamma_0\dot{\chi}\right) + \frac{1}{4}\left(\Gamma^i\omega_i^{ab} - \Gamma_0\omega_0^{ab}\right)\gamma_{ab}\chi - ma^2\bar{\chi}\chi + \frac{1}{a}\bar{\chi}\left(a_i'\Gamma^i + \dot{a}\Gamma_0\right)\chi\right] \tag{32}$$

When comparing Eqs. 31 and 29, we see that the action has developed an extra piece in the form $\frac{1}{a}\bar{\chi}\left(a_i'\Gamma^i + \dot{a}\Gamma_0\right)\chi$. Note, since $a = a(\eta)$ is not a function of the spatial component of the spacetime then $a_i' = 0$, and the extra piece is just $\frac{\dot{a}}{a}\bar{\chi}\Gamma_0\chi$, and expanding $\Gamma_0 = e_{0a}\gamma^a$ gives us $\frac{\dot{a}}{a}\bar{\chi}e_{0a}\gamma^a\chi$.

What exactly does this term represent? If we use the standard representations for the γ-matrices and tetrad coordinates for an FRW universe then it gives us a quartic interaction vertex that is proportional to $-\frac{1}{a^2}(\bar{\chi}\sigma^i\chi)^2$, where and σ^i are the usual Pauli matrices. However, we are interested in what this term does to the mass and frequency of the auxiliary spinors χ. First of all, this is not a *mass* term for the spinors. While the mass term for spinor fields in a static, flat space have the form $\bar{\chi}m\chi = m\bar{\chi}\chi$, for expanding, curved spacetimes the mass m is replaced by a spacetime dependent mass matrix $M(x)$ to give the mass term $\bar{\chi}M(x)\chi$. Here the spacetime dependence cannot explicitly come from the coordinates $\{e_a^\mu\}$ which is exactly what we have. Another way to see that this is not a mass term is to remark that the mass term $\bar{\chi}m\chi$ has only one factor of a γ-matrix (hidden in $\bar{\chi}$), whereas $\frac{\dot{a}}{a}\bar{\chi}e_{0a}\gamma^a\chi$ has two factors and thus it can never represent a mass term for the spinors. Second, this term however does affect the equations of motion. Varying the action functional with respect to the fields $(\chi, \bar{\chi}, e_a^\mu)$ give the following equations of motion:

$$\frac{\delta S}{\delta\chi} = \partial_\mu\left(\bar{\chi}\Gamma^\mu\right) + \bar{\chi}\left(\frac{\dot{a}}{a}\Gamma^0 - ma^2\right) = 0$$
$$\frac{\delta S}{\delta\bar{\chi}} = \Gamma^\mu\partial_\mu\chi + \left(\frac{\dot{a}}{a}\Gamma^0 - ma^2\right)\chi = 0$$

$$\frac{\delta S}{\delta e^\mu_a} = \bar{\chi}\gamma^a\left(\frac{1}{a}\partial_\mu a + \partial_\mu \chi\right) = 0$$

It is noted that when integrating by parts the variations of the form $\delta(\partial_\mu \chi) = \partial_\mu(\delta\chi)$ [10] we are able to disregard the boundary terms as the FRW universe is asymptotically flat, and the fields $(\chi, \bar{\chi})$ vanish at infinity. In the equations of motion for $(\chi, \bar{\chi})$, the extra terms that are proportional to $\frac{\dot{a}}{a}\Gamma^0$ make it so that the Fourier expansions of the EOM solutions develop time dependent frequencies $\omega(t)$. A time dependent frequency is problematic as it doesn't allow us to define a vacuum. Analogous to the case of scalar fields, the expansion of χ into Fourier modes includes creation and annihilation operators $(a^\dagger_k, a_k)$ which are dependent on the frequency (and through the usual dispersion relation, the momentum) of the spinor. This is problematic because time-dependent annihilation operators that annihilate the vacuum at some time don't necessarily annihilate the vacuum at another time. Since we can't define the notion of a vacuum, we can't describe fields or particles in the expanding spacetime. This also prevents us from defining an effective Wilsonian action where we integrate out high energy spinor modes to keep observables finite, being that kept low energy modes can evolve in time to become high energy modes. The same issues persists for more general p-form fields. Now, with these issues presented, instead of trying to construct a vacuum solution to the Einstein field equations (de Sitter space) with a vacuum $|\Omega\rangle$, we consider excited Glauber-Sudarshan states $|\sigma\rangle$ over supersymmetric Minkowski space $\mathbb{R}^{3,1}$ (the supersymmetric extension of Minkowski space $\mathbb{R}^{3,1}$, another vacuum solution of the Einstein field equations).

We have seen over an expanding space the bosonic fields have their mass altered while the fermionic fields don't. This means (as seen in the appendix) that the zero point energy no longer cancels out and the individual frequencies diverge. This also means we would have to perturb the system about a diverging vacuum energy. This explicitly breaks supersymmetry (additional contributions come from fluxes and branes) and so the theory does not agree with our compactifications coming from type II (super)string theory. This makes us instead consider Glauber-Sudarshan states $|\sigma\rangle$ over a super Minkowski space $\mathbb{R}^{3,1}$ which only breaks supersymmetry spontaneously (meaning that the vacuum remains supersymmetric and the zero point energy cancels).

2.3 Excited coherent states

Here we show we can compactify a type II string theory to a de Sitter universe. Recall that in previous sections we have confirmed that the massive sector of string theory which arises from compactification (the Kaluza-Klein states) have vanishing contributions to the metric and interaction diagrams under a compactification of the internal space. Furthermore, we saw that fields on an expanding geometry lead to explicit supersymmetry breaking, as well as the inability to define a vacuum, hence we cannot define an effective Wilsonian action of the theory at a given energy scale. Here we look at circumventing these problems (including the classical no-go theorems) by working over a flat super Minkowski space $\mathbb{R}^{3,1}$ and instead consider general coherent states, known as *Glauber-Sudarshan* states $|\sigma\rangle$, which expand and share the isometries of de Sitter space. Being that these states expand and have the same isometries as the de Sitter space metric, they are of interest to replace the problematic interacting de Sitter vacua $|\Omega\rangle$. The objective is to use states to compute quantum corrections to the metric *operator* of the form $\langle\sigma|\hat{g}_{\mu\nu}|\sigma\rangle$ to allow for non singular de Sitter compactifications.

What does this all mean? Did we fall down the wrong rabbit hole just as Alice did in her adventures in

Wonderland? Not quite so! To see this, first we must consider why we use coherent states at all in QFT and what are the Glauber-Sudarshan states are. Additionally, we should understand the nature of the space we work over (super Minkowski space) and what it mean for a state to share the isometries of a space? Once we figure out those two pieces of information we move onto the metric quantum corrections.

For a *free* QFT with a single bosonic DOFs $\alpha_1 \equiv \alpha$ for example, the coherent state $|\alpha\rangle$ is a shift of the free vacuum $|\alpha\rangle = \mathbb{D}_0(\alpha)\,|0\rangle = \exp\left(\alpha a_k^\dagger - \alpha^* a_k\right)|0\rangle$, where $\mathbb{D}_0(\alpha)$ is the unitary displacement operator, α is a complex number, and $(a_k^\dagger, a_k)$ are the usual creation and annihilation operators. These states are useful since they preserve the *coherence* of the quantum system, where the coherence of the system refers to the degree to which a quantum system exhibits wave-like behavior, such as interference and diffraction. Furthermore, they are the excited quantum states that most closely resemble classic states, with minimal uncertainty in position and momentum. This is useful to us as our theory must reproduce classical physics in certain limits. In the case for an *interacting* QFT however, with multiple bosonic DOFs $\{\alpha_i\} \equiv \sigma$, we instead have $|\sigma\rangle = \mathbb{D}(\sigma, t)\,|\Omega\rangle$, the Glauber-Sudarshan (GS) states. Being that the vacuum which is excited is now *interacting*, then there is an ambiguity of the displacement operator and it becomes non-unitary. For the case of interacting vacua, the structure isn't as trivial as the free vacua and so preserving unitarity of the shift operator is difficult. It is however still possible to represent it in terms of the interacting Hamiltonian over a temporal domain [11]:

$$\mathbb{D}(\sigma, t) = \lim_{T \to \infty(1-i\epsilon)} \mathbb{D}_0(\sigma, t) \exp\left(iM_p \int_{-T}^{t} dt\; \mathbf{H}_{\text{int}}\right). \tag{33}$$

Here $\mathbb{D}_0(\sigma, t)$ is a time dependent unitary displacement operator for the free vacuum, $\mathbf{H}_{\text{int}}$ is the interacting Hamiltonian of the theory, M_p is the Planck mass, and the limit is slightly in the imaginary direction $(1 - i\epsilon)$ for the same reason it for propagators: to avoid poles in the complex phase that give rise to divergences in observables. Now, for reasons which will be mentioned later, we must *uplift* to M-theory, an 11D theory. 10D type IIA and IIB string theory can then be recovered by compactifying along $(S^1, \mathbb{T}^2 = S^1 \times S^1)$, respectively. The multiplet for the fields in M-theory is given by: $(g_{ab}, C_{abc}, \psi_a)$, where g_{ab} is the metric field, C_{abc} is a 3-form axion field, and ψ_a is a Rarita–Schwinger fermion. The fields in the multiplet respectively come with DOFs $(\{\alpha_{ab}\}, \{\beta_{abc}\}, \{\gamma_a\})$. In the bosonic sector with 128 DOFs, we have a graviton g_{ab} with 44 DOFs $\{\alpha_{ab}\}$, and a three-form $\{C_{abc}\}$ with 84 DOFs $\{\beta_{abc}\}$ [12]. The fermionic sector admits fermionic condensates ψ_a which contribute 128 DOFs $\{\gamma_a\}$. Denoting all the DOFs as $\sigma \equiv (\{\alpha_i\}, \{\beta_i\}, \{\gamma_i\})$, we write the coherent product (Glauber-Sudarshan) state as [12]:

$$|\sigma\rangle \equiv \mathbb{D}(\sigma, t)\,|\Omega\rangle = \bigotimes_k \left(\sum_{f_k} \Psi^\sigma(f_k)\,|f_k\rangle\right), \tag{34}$$

where the sum is over all the fields $f_k = (\{g_{ab}(k)\}, \{C_{abc}(k)\}, \{\psi_a(k)\})$, the tensor product is over the mode momenta k, Ψ^σ is the wave function of the GS state (corresponding to the DOFs of the field f_k given by σ) which can be thought of as a product of normalized Dirac delta functions, and $|f_k\rangle$ are eigenstates of the momentum wavefunction of the GS states, in the configuration space. This is a general coherent state over the full 256 DOFs, decomposed into the GS wavefunction delta modes. Now that we have a description of the states we will use for the quantum corrections we move onto describing the isometries of the space.

Now, since type II theory is a 9+1D theory, we must compactify on a 6D manifold to recover a 3+1D theory (our de Sitter spacetime). For explicit calculations, it is useful to pick a *slicing* of de Sitter space. This is in reference to a *foliation* of de Sitter space which is an equivalence class of its submanifolds. These different submanifolds are known as the *leaves* of the foliation and in essence decompose the space into smooth hypersurfaces with different coordinates. In this case we choose coordinates that make de Sitter space look flat within a certain region. We do this by using the *flat* slicing of de Sitter space dS given by: $ds^2 = 1/(\Lambda|t|^2)\eta_{\mu\nu}dx^\mu \otimes dx^\nu$. Here t is the conformal time coordinate (instead of writing η), Λ is the cosmological constant, $\eta_{\mu\nu}$ is the usual Minkowski metric, and dx^μ is the basis living on the fibres of the cotangent bundle. A flat slicing gives us a temporal domain over which the metric is defined. The range is give by $-1/\sqrt{\Lambda} \leq t < 0$, where $t = 0$ represents late times[†]. Next, we define the full superspace we are working over as $\mathcal{M}_{10|4} = \mathbb{R}^{3,1|4} \times \mathcal{M}_4 \times \mathcal{M}_2$, where $\mathbb{R}^{3,1}$ is our super Minkowski space, and $\mathcal{M}_4 \times \mathcal{M}_2 \equiv \mathcal{M}_6$ is some non-Kähler 6D internal space (written as a product space to account for multiple scaling factors for different pieces of the internal space). The total superspace has a dimension of (10|4)D (with 4 referring to the 4 pair of Grassmanian variables) however it suffices for our analyses to look at the non-supersymmetric 10D part of the space: $\mathcal{M}_{10} = \mathbb{R}^{3,1} \times \mathcal{M}_4 \times \mathcal{M}_2$, and realize it as a Glauber-Sudarshan state. Moreover, to align with more general geometries that include warping, we define the metric as the warped geometry [11]:

$$ds^2 = \frac{1}{\Lambda H^2(y)|t|^2}\eta_{\mu\nu}dx^\mu \otimes dx^\nu + H^2(y)\left[F_1(t)g_{\alpha\beta}dy^\alpha \otimes dy^\beta + F_2(t)g_{mn}dy^m \otimes dy^n\right] \tag{35}$$

Here $H(y)$ is the *warp factor* that depends on the internal space coordinates $\{y^m, y^\alpha\}$ which mixes the DOFs of the internal and external space, and $\{x^\mu\}$ are the coordinates of the external space $\mathbb{R}^{3,1}$. Furthermore, $(F_1(t), F_2(t))$ are scaling factors of the different internal subspaces $(\mathcal{M}_4, \mathcal{M}_2)$, respectively, equipped with metrics $(g_{\alpha\beta}, g_{mn})$. However, when computing quantum corrections such as $\langle\sigma|\,\hat{g}_{\mu\nu}\,|\sigma\rangle$, we must compute path integrals over the flat metric to reproduce the above metric. Path integrals require a full description of the system's action and as of yet there are no well-defined action for type IIB string theory [13]. This makes us consider the uplift of type IIB string theory to M-theory, an 11D string theory. The action of lifting can be thought as the reverse process of compactification whereby different compactifications of M-theory give us the different type II string theories (we can recover type IIA string theory by compactifying on S^1, and type IIB on $T^2 = S^1 \times S^1$). Thus, we work in the framework of M-theory, compactify once to get type IIA string theory, and compactify a second time to get the type IIB theory. In this case the multiplet for M-theory is $(g_{MN}, C_{MNP}, \psi_M)$, where g_{MN} is the metric field over the full 11D space, C_{MNP} is a 3-form axion field, and ψ_M is a vector-spinor (also known as a gravitino). This 11D space is the 11D supermanifold: $\mathcal{M}_{11} = \mathbb{R}^{3,1} \times \mathcal{M}_4 \times \mathcal{M}_2 \times \frac{\mathbb{T}^2}{\mathbb{Z}_2}$. Here $\mathbb{R}^{3,1}$ is our super Minkowski space, $\mathcal{M}_4 \times \mathcal{M}_2 \equiv \mathcal{M}_6$ is the internal space, and $\frac{\mathbb{T}^2}{\mathbb{Z}_2}$ is the extra 11th direction where $\mathbb{Z}_2$ is the orbifold action, and $\mathbb{T}^2 = S^1 \times S^1$ is the 2-torus. As with type II string theory, the full supermanifold is 11D but we will look at a non-supersymmetric state in 11D. We define coordinates of the space as: $(x^\mu, x^\nu) = (x^0, \ldots, x^3)$, $(y^M, y^N) = (x^4, \ldots, x^9)$, $(\omega^a, \omega^b) = (x^{10})$, for the spaces $(\mathbb{R}^{3,1}, \mathcal{M}_6, \frac{\mathbb{T}^2}{\mathbb{Z}_2})$, respectively. Based off of [13], we can write the of the space in M-theory as the following:

[†]Due to the restriction of the spacetime domain, **the GS state remains coherent over this same time domain**.

$$ds^2 = \left(\frac{g_s}{\mathrm{HH}_o}\right)^{-8/3} \tilde{g}_{\mu\nu} dx^\mu \otimes dx^\nu + \left(\frac{g_s}{\mathrm{HH}_o}\right)^{-2/3} \left[\mathrm{F}_1 \left(\frac{g_s}{\mathrm{H}_1}\right) \tilde{g}_{\alpha\beta} dy^\alpha \otimes dy^\beta + \mathrm{F}_2 \left(\frac{g_s}{\mathrm{H}_1}\right) \tilde{g}_{mn} dy^m \otimes dy^n \right] + \left(\frac{g_s}{\mathrm{HH}_o}\right)^{4/3} \tilde{g}_{ab} d\omega^a \otimes d\omega^b. \tag{36}$$

Here $g_s = \sqrt{\Lambda}|t|H(y)$ is the type II string coupling constant where Λ is the cosmological constant, t is the conformal time, and $H = H(y)$ is the warp factor. Furthermore, $H_0 = H_0(x) \propto g_{33}(x)$, $H_1 = H_1(x, y) = H(y)H_0(x)$, $(\tilde{g}_{\mu\nu}, \tilde{g}_{MN}, \tilde{g}_{ab})$ are the metrics (up to some factors of H and H_0) of the spaces $(\mathbb{R}^{3,1}, \mathcal{M}_4 \times \mathcal{M}_2, \frac{\mathbb{T}^2}{\mathbb{Z}_2})$ with cotangent bases $(dx^\mu, (dy^\alpha, dy^m), d\omega^a)$, and F_i are the scaling factors for the different pieces of the internal space which depend on the temporal scaling factor $a(t)$. Now that we have an expression for the space we are working with, and have defined the Glauber-Sudarshan states $|\sigma\rangle$, we move with computing the quantum correction to the metric operator $\langle\sigma| \hat{g}_{\mu\nu} |\sigma\rangle$ which is normalized by $\langle\sigma|\sigma\rangle$. Plugging in the definitions $|\sigma\rangle = \mathbb{D}(\sigma, t) |\Omega\rangle$, and dividing the numerator and denominator by $\langle\Omega|\Omega\rangle$ we see that these are both path integrals (functional integrals over all possible evolutions of the fields) over the M-theory multiplet in 11D [13]:

$$\langle\sigma| \hat{g}_{\mu\nu} |\sigma\rangle \equiv \langle\hat{g}_{\mu\nu}\rangle_\sigma = \frac{\displaystyle\int \mathcal{D}[g_{MN}] \wedge \mathcal{D}[C_{MNP}] \wedge \mathcal{D}[\psi_M] \wedge \mathcal{D}[\bar{\psi}_N]\, e^{iS}\, \mathbb{D}^\dagger(\sigma, t) g_{\mu\nu}(x) \mathbb{D}(\sigma, t)}{\displaystyle\int \mathcal{D}[g_{MN}] \wedge \mathcal{D}[C_{MNP}] \wedge \mathcal{D}[\psi_M] \wedge \mathcal{D}[\bar{\psi}_N]\, e^{iS}\, \mathbb{D}^\dagger(\sigma, t) \mathbb{D}(\sigma, t)}. \tag{37}$$

Here $\mathcal{D}[A_{M\ldots N}] \sim \prod_{M,\ldots,N} dA_{M\ldots N}$ are the path integral field measures (where $A_{M\ldots N}$ represents the different fields of the multiplet which are integrated over), S is the total action of the system, $\mathbb{D}^\dagger(\sigma, t)$ is the non-unitary shift operator, and $g_{\mu\nu}(x)$ is the metric of the external Minkowski space $\mathbb{R}^{3,1}$ (which when including the supersymmetric part of the supermanifold becomes super Minkowski space $\mathbb{R}^{3,1}$). It is noted that the measures associated with *ghosts* $\{c_i, \bar{c_j}\}$ (fields with negative norm used to maintain gauge invariance) are suppressed. Now this path integral is much too complex at this level of generality with Grassmanian integrals, so we pick three representative sample scalars $(\varphi_1, \varphi_2, \varphi_3)$ for each of the DOFs sectors (NS-NS, R-R, NS-R). Theses scalars fix the DOFs to $\sigma \equiv (\varphi_1, \varphi_2, \varphi_3)$, and so for the DOFs φ_1 that contribute towards the metric operator correction we have the replacement: $\langle\hat{g}_{\mu\nu}\rangle_\sigma \to \langle\varphi_1\rangle_\sigma$. Once fixed, the path integrals in the numerator and denominator of $\langle\varphi_1\rangle_\sigma$ fall under a class of path integrals that maybe computed using *nodal* diagrams due to the shifted vacuum structure of the Glauber-Sudarshan state. These nodal diagrams are general interaction diagrams connected by *nodes*, which are configurations of incoming momenta. Using the amplitudes of the nodal diagrams $\{\mathcal{A}_s\}$ it is possible to express the quantum correction as the following [12]:

$$\langle\varphi_1\rangle_\sigma = \frac{\mathrm{TLN} + \displaystyle\sum_{n,\ldots,s} c_{mnpqrs}\, \mathcal{N}^{(1)}_{nmp}(k; q) \otimes \mathcal{N}^{(2)}_{nmp}(l; r) \otimes \mathcal{N}^{(3)}_{nmp}(f; s)}{\mathrm{TLD} + \displaystyle\sum_{n,\ldots,s} c_{mnpqrs}\, \mathcal{N}^{(1)'}_{nmp}(k; q) \otimes \mathcal{N}^{(2)}_{nmp}(l; r) \otimes \mathcal{N}^{(3)}_{nmp}(f; s)} \tag{38}$$

Here (TLN,TLD) are the tree-level contributions to the numerator and denominator path integrals, respectively. The loop contributions are captured by the summation over the coupling constants c_{nmpqrs} and $\{\mathcal{N}^{(i)}_{nmp}(a; b)\}$ are a diagrammatic series of amplitudes of different nodal diagram classes. Here $\mathcal{N}^{(i)}_{nmp}(a; b)$

is associated with the interaction of fields of the form $\varphi_i^b(a)$, where b is an integer, a is the momentum of the incoming fields, and $i = 1, 2, 3$ are the different scalars which represent the DOFs of the multiplet in M-theory. Furthermore, the prime in $\mathcal{N}^{(1)'}_{nmp}(k; q)$ refers to interaction diagrams without sources. These nodal diagrams diverge with structure of the Gevrey kind (meaning they diverge factorially) and require Borel resummation to restructure the divergence into non-perturbative solitonic corrections. Using the nodal amplitudes and Borel re-summation gives the following correction to the metric operator [13]:

$$\langle \varphi_1 \rangle_\sigma = \sum_{\{s\}} \left[\frac{1}{g_{(s)}^{1/l}} \int_0^\infty dS \exp\left(-\frac{S}{g_{(s)}^{1/l}} \right) \frac{1}{1 - \mathcal{A}_{(s)} S^l} \right]_{P.V} \int_{k_{\rm IR}}^{\mu} d^{11}k \; \frac{\bar{\alpha}_{\mu\nu}(k)}{a(k)} \; \mathbf{Re} \left(\psi_k(X) e^{-i(k_0 - \bar{\kappa}_{\rm IR})t} \right), \tag{39}$$

where $\{s\}$ is the set of interactions, $g_{(s)}$ is the set of coupling constants, S parametrizes an axis in the Borel plane, $\mathcal{A}_{(s)}$ is the amplitude of all possible nodal diagrams, $P.V$ is the principal value of the integral over S, $(k_{\rm IR}, \bar{\kappa}_{\rm IR})$ are IR scales, $\bar{\alpha}_{\mu\nu} = \alpha_{\mu\nu}/V$ is a 2-form normalized by the volume V of the space $\mathcal{M}_{11}$, $a(k) = k^2/V$ are, and $\psi_k(X)$ is the spatial wavefunction of the Glauber-Sudarshan state over the space $\mathcal{M}_{11}$ (projecting the GS state $|\sigma\rangle$ into the coordinate space of $\mathcal{M}_{11}$). It turns out the first piece of the form $\sum[\dots]$ exactly corresponds to the inverse of the cosmological constant (at a given IR scale κ) [12], which is defined over the same time domain as the Glauber-Sudarshan state ($-\frac{1}{\sqrt{\Lambda}} < t \leq 0$):

$$\frac{1}{\Lambda^\kappa} \equiv \sum_{\{s\}} \left[\frac{1}{g_{(s)}^{1/l}} \int_0^\infty dS \exp\left(-\frac{S}{g_{(s)}^{1/l}} \right) \frac{1}{1 - \mathcal{A}_{(s)} S^l} \right]_{P.V}, \quad -\frac{1}{\sqrt{\Lambda}} < t \leq 0. \tag{40}$$

This is known as the *integral form* of the cosmological constant Λ^κ, and is **positive definite** over the flat slicing of the temporal domain. To recover the result for type II string theory from M-theory, one need only compactify along a compact direction, which leaves the above form of the cosmological constant unchanged. This shows us that using the Glauber-Sudarshan states (which have the isometries of Minkowski space and preserve supersymmetry) gives us the correct sign for the cosmological constant. Recall the issues of the classical no-go theorems and field theories on expanding spacetimes are not present in the formulation of a coherent state over the Minkowski background. Additionally, the Glauber-Sudarshan states obey the energy conditions mentioned at the beginning of section 2 [12]. Here the cosmological constant functional expression comes from the quantum correction of the metric and it having the correct positive sign means we *indeed* have compactifications to de Sitter dS_4 space from type II string theory. Hence, we can consider de Sitter space as a candidate of our universe as coming from a type II string theory compactifications.

3 Discussions and conclusion

Classical no-go theorems prevent non-singular compactifications to de Sitter space in type II superstring theory. This prevents us from having a description of supergravity or have vacuum energies which produce the right sign, given by the interacting vacua $|\Omega\rangle$. Furthermore, we see that fields defined over an expanding, curved spacetime develop time dependent frequencies and thus prevent us from defining an effective Wilsonian action at a given energy scale. This comes from the ambiguity of a time-dependent interacting vacuum $|\Omega(t)\rangle$. Further complications come from the fact that the space must preserve certain energy conditions, and that the g_{00} component of the metric cannot diverge as we approach flux source singularities on the spacetime manifold.

This pushes us to instead consider a class of general coherent states, known as Glauber-Sadarshan states $|\sigma\rangle$ over super Minkowski space $\mathbb{R}^{3,1}$ which have the desired de Sitter isometries and does not run into the issues mentioned above. It is found that we can define these states over a given temporal domain via the non-unitary shift operator $\mathbb{D}(\sigma, t)$ which excites the interacting vacua to give the GS states which preserve the isometries of super Minkowski space $\mathbb{R}^{3,1}$. This allows us to compute the metric operator quantum corrections $\langle\sigma|\hat{g}_{\mu\nu}|\sigma\rangle$. From the quantum correction, a piece is identified with the cosmological constant Λ^{κ} (at a given IR scale κ) and is positive definite, meaning it has the correct sign. This means we can have consistent compactifications to de Sitter space dS_4 in type II string theory and thus de Sitter space can be used as a candidate for our universe where the vacua are instead Glauber-Sadarshan states $|\sigma\rangle$.

4 Acknowledgements

I would like to thank Simon Caron-Huot, Brent Pym, Evan McDonough, Mathieu Boisvert, Clément Virally, Guillaume Payeur, Bohdan Kulinich, and David leNir for our discussions related to string theory, algebraic topology, and string cosmology. I would also like to thank Jonathan Eid, Alexander Freire, Mary Letey, Alexander Kroitor, and Hans Hopkins for reading and giving feedback on previous drafts of this paper. Additionally I would like to thank Keshav Dasgupta (among many other things) for teaching a superstring theory course over 8 months which helped immensely in catching up to speed and understanding the framework of string theory, prior to which (and hence prior to the thesis) I knew nothing of.

5 Appendix: Pertinent preliminaries

It is important to have some understanding of the framework we use (superstring theory) to have a better picture of the analysis being done. In the first section of the preliminaries we start by motivating why we need frameworks beyond QFT when also considering gravity. Thereafter, we look at which fields (and their cotangent form duals) can be defined on the spacetime manifold, giving us a description of the different fundamental particles that can exist in our universe. After that we look at generalization of point particles (branes) which are needed for the world sheet description of string theory, whereby the fields we define that live higher dimensions require higher dimensional objects objects to be defined on (including orientfold planes). These higher dimensional objects (fields, branes, orientfold planes) act as flux sources in the theory. Next, we look at a characteristic feature in superstring theory which is supersymmetry, an extra symmetry of the action which allows us to alleviate certain problems in non-supersymmetric field theories. Finally, we look at compactification of manifolds which will play a big role in computations, and is relevant to the algebraic and topological structure of string theory. This allows us to separate and control the extra dimensions that result from string theory, and recover a 3+1 dimensional spacetime in which we currently reside.

5.1 Frameworks beyond QFT

Before proceeding in the framework of supersymmetric string theory, we must first look at why such a framework is required at all. Recall that QFT was a necessary construction as quantum mechanics fails to describe the interactions of particles with different characteristics such as colour/adjoint representation indices, chirality, and helicity (or phenomena such as vacuum polarization for example). Quantum field theory nevertheless has issues, one such being the failure of perturbation theory in higher order Feynman diagrams of interacting particles, in which the amplitude for the n-th loop diagram scales as $n!$[†] and diverges [14]. A more infamous and relevant problem to us is the non-renormalizability of gravity. A typical QFT has an effective Wilsonian action defined at a certain energy scale. This is done by imposing a low energy/IR cutoff which acts as a regularization scheme for the integrals over the high energy modes. The process of integrating out high energy modes (known as Wilsonian integration) generates a renormalization group flow where the theory may flow to a conformal/Gaussian fixed point in the coupling constant phase space. Once the iteration process is done we are left with an effective action at a given energy/UV scale where there are no longer any irrelevant operators. When trying to couple the theory to gravity, however, there is an anomalous mixing between the UV and IR cutoff values and so we cannot reconcile the discrepancy between the physics at the different scales. To explicitly see the shortcoming of combining gravity with QFT, consider the *Einstein-Hilbert* action, the simplest action we can construct from the metric $g_{\mu\nu}$ and the Ricci curvature form $R_{\mu\nu}$, which gives rise to the vacuum Einstein field equations:

$$S = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,(g^{\mu\nu}R_{\mu\nu}) = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}R. \tag{41}$$

Here G is the Newton gravitational constant, $\sqrt{-g} = \sqrt{-\det g_{\mu\nu}}$ is root determinant of the metric used to make the integral invariant under Lorentz transformations, and R is the Ricci scalar which captures the curvature of the spacetime. To couple this to a quantum field theory, we note the relevant coupling of the quantum theory to gravity is $1/M_{pl}$ [15], where M_{pl} is the Planck mass. To construct interactions of perturbing curvature of the spacetime in the form of gravitons, consider small perturbations around a flat Minkowski space $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{pl}}h_{\mu\nu}. \tag{42}$$

Here $h_{\mu\nu}$ is a rank-2 symmetric tensor who's components are small such that $h_{\mu\nu} \ll 1$. Plugging this perturbation into the action defined above, and identifying the Newton constant as $G = 1/(8\pi M_{pl}^2)$ we end up with:

$$S = \int d^4x\,\left[(\partial^\mu h_{\mu\nu})^2 + \frac{1}{M_{pl}}h(\partial^\mu h_{\mu\nu})^2 + \frac{1}{M_{pl}^2}h^2(\partial^\mu h_{\mu\nu})^2 + \ldots\right] = \sum_{k=0}^{\infty}\int d^4x\,\left[\left(\frac{h}{M_{pl}}\right)^k(\partial^\mu h_{\mu\nu})^2\right]. \tag{43}$$

Here $\partial^\mu = g^{\mu\nu}\partial_\nu$ is the regular partial derivative ∂_ν contracted with the metric to raise the index, and h are the graviton fields. This leads to an infinite number of gravitational interaction vertices in the form of graviton fields h^k, which become strongly coupled at high energies and are irrelevant with respect to the normalization group flow.

To see the issue with this, we can compare this with the theory of quantum electrodynamics (QED) where we get a vertex of the form $g\bar{\psi}\psi\bar{\psi}\psi$ where $(\psi, \bar{\psi})$ are fermionic field spinors and $g \ll 1$ is the relevant coupling constant of the interactions. It turns out that this a non-renormalizable vertex at high energies and gives unphysical diverging results. In QED we can attempt to resolve this problem by including extra DOFs

[†]Current implementations to fix this include Borel resummation which demonstrates the non-perturbative structure of solitonic corrections.

in the theory and recover a vertex in the action of the form $S \sim g\bar{\psi}\psi A - m^2 A^2$ where A is a field describing new massive particles added in the theory. This vertex turns out to be renormalizable. This essentially adds massive virtual particles which mediate the interactions of the defined particles, but for our case with gravity, we would need to add an infinite amount of massive virtual particles to control the behaviour of the infinite amount of graviton interactions. A theory where we require to add an infinite amount of new DOFs and information to describe the behaviour of fields does not lead into a consistent QFT with point particles. We thus turn toward the framework of string theory where we replace point particles by strings and don't run into the conflicts mentioned above. The reasons for selecting a supersymmetric version of string theory, known as superstring theory, is due to the fact that non-supersymmetric theories develop tachyons. These tachyons are particles with imaginary mass that propagate faster than light, which indicate the theory is being defined over the wrong vacuum. The inclusion of super/fermionic strings (and hence supersymmetry) does not run into this issue. We now turn to defining the fields on the spacetime manifold which tells us which particles can exist in the theory.

5.2 Field representations

It is informative to define which fields and forms can exist on our spacetime as this tells us exactly what kind of fundamental particles can exist in our universe. We can switch between fields and their dual forms via a *musical* isomorphism $\sharp$ that maps between the cotangent and tangent bundles for a given space M: $\sharp : T^*M \to TM$. One way to constrain such objects on the manifold is by looking at its topological properties via the homology and cohomology groups. This tells us which closed forms (and thus harmonic forms via Hodge's theorem of form decomposition) and cycles can live on the manifold up to some equivalence class as a consequence of the topology. This is given by the Betti and Hodge numbers (dimensions of the real and complex cohomologies), and the topological properties of the space. The study of complex manifolds this way is a very fascinating topic but shall only come up again in our discussions should we deal with the Calabi-Yau manifolds. An excellent review of complex manifolds may be found in the lectures by Candelas [6]. An alternate approach which we follow is to consider the symmetry group representations. A group (specifically a Lie group) has an infinite amount of possible representations each that give different information. The fields that can exist in our theory are given by the tensor representation of the group, which are finite tensor products of the group's fundamental representation (given by matrices or roots of the associated Lie algebras). These fields/tensor representations transform under other representations of the group (either the operator or adjoint representation). We look at constructing the fields of the theory through the tensor representation and thereafter describe the different multiplets of fields allowed. This is analogous to the construction of singlet and triplets in spin-$\frac{1}{2}$ systems in quantum mechanics or the construction of multiplets in particle physics via the Young Tableaux decomposition.

For our case of superstring theory, we have two sectors of tensor representations: the bosonic sector and the fermionic sector, which are analogous to the bosonic and fermionic fields in QFT. Bosonic strings are 10-dimensional and thus it would be a natural assumption to assume the Lorentz group under which it transforms is $SO(10)$. This is not the case, however, as for a 10 dimensional momentum vector k^μ, we can perform a Lorentz transformation to make all the components vanish except for the first which is required for the mass of the state. The remaining 9 null components are trivially invariant under $SO(9)$ and so the symmetry group is $SO(9)$. Massless states on the other hand have one less DOFs as they have no mass and instead transform under $SO(8)$. We will take the constraint that all fields in the theory must have the same mass (as is required for supersymmetry), and take the masses to vanish to reproduce the fields appearing in the massless sector (low energy / classical limit) of string theory: supergravity.

To construct the tensor representations (and thus the fields) we will make use of roots and weights of the symmetry groups' associated Lie algebras. Consider a Lie algebra $\mathfrak{g}$ of a symmetry group $SO(2n)$ for $n \in \mathbb{Z}$, over some field $\mathbb{F}$. The algebra has itself a set of subalgebras $\{\mathfrak{g}_j\} \subset \mathfrak{g}$, and for one such subalgebra $\mathfrak{g}_s$ we can choose a basis of commuting Hermitian generators H^i where $i = 1, \ldots, \dim(\mathfrak{g}_s)$. The remaining generators, that is the generators of $\mathfrak{g}$ that do no span $\mathfrak{g}_s$, are given by E^α for $\alpha = 1, \ldots, 2n - \dim(\mathfrak{g}_s)$. The two sets of generators are diagonalized in such a way that they obey the commutation relation [16]:

$$[H^i, E^\alpha] = \alpha^i E^\alpha, \tag{44}$$

where α^i is an n-dimensional vector called a *root* of the algebra. These roots are of particular interest in QFT since if the first non-zero component of the root α^i is positive, then E^α is a *creation operator*, while if negative it is an *annihilation operator*. These roots will act as the fundamental representations of the group from which we take tensor products to construct the tensor representations. These tensor representations are exactly the different fields that exist on the spacetime which are collected in tuples by which group they transformed under (grouped up into multiplets of fields). The n-component root vectors in a theory with a symmetry group $SO(2n)$ has the form [16]:

$$(\ldots, \pm 1, \ldots, \pm 1, \ldots), \tag{45}$$

where all other entries are zero. All non-zero entries are called the *weights* of the algebra, which classify the different roots. That being said, what are the roots in our case? The two roots we will consider come from excitations of open string states for bosonic and fermionic vacua, respectively. Since we are dealing with $SO(8)$ in a 10-dimensional space, the roots will have $8/(10-8) = 4$ components. The bosonic root is the first excited state of an open bosonic string (in what is called the Neveu-Schwartz sector) which is the root $\mathbf{8_v} \equiv v = (\pm 1, 0, 0, 0)$ [17]. On the other hand, the fermionic root is the first excited state of an open fermionic string in the Ramond sector which gives the root $\mathbf{8_f} \equiv f = (-\frac{1}{2}, -\frac{1}{2}, 0, 0)$ [‡]. We can use this to construct the allowed fields of the theory in the form of multiplets via the decomposition of the tensor products of the roots. An example of such is what kind of fields we can construct from combining two copies of the root of the Neveu-Schwartz sector:

$$v \otimes v = (\pm 1, 0, 0, 0) \otimes (\pm 1, 0, 0, 0) = (\pm 2, 0, 0, 0) \oplus (1, 1, 0, 0) \oplus (0, 0, 0, 0). \tag{46}$$

The above is a matrix, where the components on the right are different sub-matrices of the matrix. Since all the fields are grouped up in the argument of the matrix we call it a multiplet of the fields. It turns out that we identify the fields produced with a symmetric rank-2 tensor (metric) field $g_{\mu\nu}$, an antisymmetric rank-2 tensor field $B_{\mu\nu}$, and a rank-0 tensor/scalar field φ (also referred to as a dilaton), which gives us a decomposition of allowed fields:

$$v \otimes v = g_{\mu\nu} \oplus B_{\mu\nu} \oplus \varphi. \tag{47}$$

[‡] Here the chirality of the fermion is determined by if the signs on the non-trivial components are the same or are opposite.

These multiplets are usually instead written in the form $v \otimes v = (g_{\mu\nu}, B_{\mu\nu}, \varphi)$. We can continue this process by permuting the combination of the two roots to get the different multiplets of the theory (for the two most common considerations of string theory: type IIA and type IIB superstring theory) [18]:

$$\left.\begin{aligned} v\otimes v &= (g_{\mu\nu}, B_{\mu\nu}, \varphi) \\ f_o \otimes f_o &= (A_\mu, C^{(3)}_{\mu\nu\rho}) \\ v \otimes f_o &= (\chi, \xi_\mu) \end{aligned}\right\}\text{Type IIA} \qquad \left.\begin{aligned} v\otimes v &= (g_{\mu\nu}, B_{\mu\nu}, \varphi) \\ f_e \otimes f_e &= (C^{(0)}, C^{(2)}_{\mu\nu}, C^{(4)}_{\mu\nu\rho\sigma}) \\ v \otimes f_e &= (\psi, \chi_\mu) \end{aligned}\right\}\text{Type IIB}$$

Here f_o and f_e refer to an odd and even number of fermions, respectively, and $C^{(n)}_{\mu\ldots\rho}$ are known as axion fields. We see both theories admit a metric field, an antisymmetric field, and a scalar field, as is given by $(g_{\mu\nu}, B_{\mu\nu}, \varphi)$. We also see that type IIA admits odd forms $(A_\mu, C^{(3)}_{\mu\nu\rho})$ while IIB admits even forms $(C^{(0)}, C^{(2)}_{\mu\nu}, C^{(4)}_{\mu\nu\rho\sigma})$. This is noteworthy as it will restrict the dimensionality of the different branes that give charge in each theory (more on this in the branes section). Additionally, type IIA admits non-chiral fermions (χ, ξ_μ) while IIB admits chiral fermions (ψ, χ_μ). It is noted that the usual way to call these multiplets are the NS-NS (Neveu-Schwartz) sector for $v \otimes v$, the R-R (Ramond) sector for $f_i \otimes f_i$, and finally NS-R for $v \otimes f_i$. In both theories, the NS-NS and R-R sector have 128 bosonic DOFs while the NS-R sector has 128 fermionic DOFs, for a total of 256 DOFs in each theory. The DOFs of a string theory never changes under a compactification and allows us to count how many of each fields exist after compactifying from type IIA/IIB to a lower dimensional theory for example. Furthermore, the DOFs are exactly the same due to supersymmetric construction of the theory (more on this in the supersymmetry section). It is noted that all above fields are Abelian and non-Abelian fields arise from stacking multiple coincident branes together.

As mentioned before, supersymmetry requires all the masses of the fields to be the same, which is evident in the massless sector of the theory to which we have just constructed above for $SO(8)$. The massless sector (or low energy limit) of string theory is known as supergravity. The massive states of the theory (such as Kaluza-Klein states) will be found in following sections through compactification. These constructed multiplets (and thus the fields they contain within them) tell us what type of particles exist in our universe in the string theory model. Much like in QED where the gauge field A_μ gives charge to point particles, these fields naturally charge higher dimensional objects which can be thought as generalizations of point particles. These generalizations are known exactly as *branes* and we dive into their properties in the following section.

5.3 Branes and orientfolds

Above we have shown which fields that exist within our theory which are flux sources (energy flux through a given surface) that contribute to the stress tensor in the Einstein field equations. These energy sources relate directly to the conditions which restrict the existence of dS_4 compactifications. Another energy source which contributes to these conditions are extended, charged objects which comprise of (anti) branes and (anti) orientfold planes. We build up the motivation for including such objects in our theory in this section.

The spacetime on which we define a theory is given by Lorentzian manifold with a metric signature of $(1, n-1)$ for an n-dimensional space admitting one temporal dimension, hence the 1 in the beginning of the signature. It is informative to consider a range of metrics on this manifold that gives rise to different properties (and thus physics) such as the curvature of the space, whether or not it expands, the sign of the cosmological constant, etc. One such property of interest is the topology of the manifold which constrains what fields and cycles (boundary submanifolds) can be defined on it and hence tells us what fundamental objects exist in our universe. In the search for a unified theory combining QFT and general relativity, certain spaces give rise to fields with much higher dimensionanilty than the current universe we live in, such as the 10-dimensional $SO(8)$ spinor mentioned in the previous section. Such an object must live in a higher dimensional space as is characteristic to string theory. This might seem problematic as we live in a 3+1 dimensional spacetime, but the extra dimensional degrees of freedom are defined over a separated product space which is taken to vanish as a shrinking compact space, also known as compactification. To illustrate, a 10-dimensional theory would have 4 of its dimensions describing our universe, while the remaining 6 dimensions are a compact space which vanishes in the compactifying limit to leave us with a 4D theory. We can constrain certain properties of the 6-dimensional space[‡], which tell us it must be a complex Calabi-Yau manifold. When not trying to directly recreate the results of the current universe we live in, these higher dimensional theories require corresponding higher dimensional objects to make sense of the notion of charge. These higher dimensional charged objects are known as *branes* and can be thought as the generalization of point particles. In topological string theory these branes are special Lagrangian submanifolds, which are part of a derived category of coherent sheaves of Calabi-Yau manifolds. This is very interesting but won't have much use in this paper. The reader however is strongly encouraged to look into this fascinating study given in Aspinwall's textbook [19].

Moving away from topological string theory, we look at branes in terms of forms. Recall in QED, there is a perturbative coupling term in the action of the form $e\int A = e\int dx^\mu A_\mu$, where e is the dimensionless interacting coupling of the theory which also represents the usual electric charge in electrodynamics, A_μ is the gauge field (or vector bundle connection), and dx^μ is the differential form basis living on the cotangent space. We say the charge of the point particles in the theory is given by the 1-form $A = A_\mu dx^\mu$ which we integrate over the spacetime manifold. We now instead turn to a rank p-tensor $A_{\mu_1}\cdots_{\mu_p}$ which are components of an associated p-form:

$$A_p \equiv \frac{1}{p!} A_{\mu_1}\cdots_{\mu_p}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}. \tag{48}$$

Here $\wedge$ is the usual wedge product between forms (antisymemtrization of the tensor product), and the factorial normalization is to account for the different ways we can recover the same p-form via the antisymmetric properties of $\wedge$. This p-form transforms under a generalized gauge transformation of the form [20]:

$$A_p \longrightarrow A_p + d\Lambda_{p-1}, \tag{49}$$

where d is the usual exterior derivative and Λ_{p-1} is a $(p-1)$-form serving as the higher dimensional parameter for gauge transformations. Much like how the objects that are naturally charged under the 1-form $A_1 \equiv A_\mu dx^\mu$ are the point particles, *p-branes* are the extended objects of p-spatial dimensions that charged under $(p+1)$-forms A_{p+1}. To gain some intuition, $p = 0$ branes corresponds to the points particles, while $p = 1$ branes corresponds to strings as so forth. The natural interacting coupling in the action that gives rise to the charge of these objects (branes) has the form:

[‡] Such constraints include compactness to have a finite Newton constant G, it being Ricci flat, and having an integrable complex structure. Such constraints are covered in the lectures by Candelas, mentioned in the previous section.

$$iQ_p \int_\Sigma A_{p+1} = iQ_p \int_\Sigma dx^{\mu_0} \wedge \cdots \wedge dx^{\mu_p} A_{\mu_0} \ldots A_{\mu_p}, \tag{50}$$

where Q_p is the electric charge density of the brane, Σ is the $(p+1)$-dimensional world volume of the brane, and in this case we take the indices of the p-form to start at μ_0 for $p = 0$. Often times in string theory the tension of the brane is mentioned in place to its charge. This correspondence is through their mass per unit volume, a relationship that is straightforward using natural units.

Furthermore, to complete the theory we also require magnetically charged objects. This comes naturally from the Montonen–Olive duality (a strong-weak duality much like S-duality) of the fields seen in Maxwell's equations. To get the magnetic analogue of the p-brane we first define the gauge invariant field strength form $F_{p+1} = dA_p$ which satisfies the free Maxwell equations $d \star F_{p+1} = 0$, where $\star$ is the Hodge star operator. To get the magnetic analogue, we dualize the field strength by applying the Hodge star and write is as an exact form of another form (where we use the notation $\star B \equiv \tilde{B}$)[20]:

$$\star dA_{p+1} = \star F_{p+2} = \tilde{F}_{D-p-2} = d\tilde{A}_{D-p-3}. \tag{51}$$

Here the change in indices from the second equality to the third is from the definition of the Hodge star operator $\star : \Omega^p(M) \to \Omega^{D-p}(M)$, where $\Omega^p(M)$ is the collection of differential p-forms on a D-dimensional space M. Thus, we see from above that the objects that are magnetically charged under $\tilde{A}_{D-p-3}$ are $(D-p-4)$-branes, with analogous magnetic charge densities $\tilde{Q}_{D-p-4}$. Thereafter, it turns out that the end points of open strings are confined to hyperplanes which are these same branes. The endpoints of strings obey Dirichlet boundary conditions and thus the branes over which they are fixed to are known as *D-branes*. The electrically and magnetically charged branes also gives rise to a generalization of Dirac quantization known as DNT quantization. It is important to note that branes are simultaneously charged by different independent sources. The presence of open strings ending on the branes give rise to electric and magnetic charge in the form of gauge fields A_μ living on the surface of the brane. Furthermore, branes are charged by the R-R fields from the previous section, which give rise to fields that are orthogonal/transverse to the brane. Much like how colour charge and electric charge are independent in QCD, the charge from the open strings and R-R fields are also independent. The existence of branes moreover gives rise to corrections in mass spectra of the states in the theory in the form of an effective action. When N D-branes are spatially coincident (each with a gauge field defined over it with a $U(1)$ symmetry) it gives rise to a $SU(N)$ non-Abelian gauge theory. This is known as the *enhancement* of gauge symmetry. There are also notions of M-branes and F-branes occurring in M-theory and F-theory respectively, as well as NS-branes, but they won't be motivated here for brevity. Recalling from the previous section that type IIA string theory admits odd forms (R-R forms/fields) while type IIB admits even forms, this gives rise to odd and even dimensional branes for each theory respectively (a Dp-brane is $p+1$ dimensional). Explicitly, type IIA admits $D0, D2, D4, D6, D8$ branes for the R-R sector and $F1$, $NS5$ branes for the NS sector. There is no NS-R sector as fermionic fields do not give rise to charged branes. Moreover, type IIB admits $D1, D3, D5, D7, D9$ branes for the R-R sector and $F1$, $NS5$ branes for NS sector. Now that we have defined the branes in our theory, where are the strings? The strings, which are the most fundamental objects of the theory (much like how in QFT the most fundamental objects are fields which give rise to particle states), are precisely the $D0$, $D1$, and $F1$ branes. $F1$-branes are the fundamental bosonic/fermionic closed/open strings that are excitations of bosonic/fermionic vacua whose endpoints are fixed to the Dp-branes. $D0$-branes are open strings that end on itself which gives rise to point particle like dynamics. Finally, $D1$-branes are strings that have other open strings ending on it. Being that strings are special cases of branes, we should instead call the theory *brane theory*, but branes themselves cannot be dynamic without strings ending on them which ultimately give rise to the fields occurring in our theory.

Branes are dynamic objects which are characterized by the different dynamic open string states defined over them. We may also have higher dimensional objects through which branes propagate: orbifolds and orientfolds. In mathematics, orbifolds are generalizations of manifolds which are locally diffeomorphic to a quotient of $\mathbb{R}^n$ and some finite group Γ given by $\mathbb{R}^n/\Gamma$ [21]. Furthermore, orbifolds/orientfolds are loci meaning the set of points they contain satisfies specific conditions/equations which precisely fix their position in space. In the context of string theory, orbifolds are objects that are written as orbit spaces M/G, where M is some manifold and G is an isometry group corresponding to M. If an open string on a D-brane has a left and right moving mode defined on it, we can perform an orientation reversal (orientfold) transformation Ω, which switches the two. If we project out the string states which are not invariant under this discrete symmetry (non-eigenstates), we get what is known as gauging Ω or orientfolding [22]. The gauging of the reflection operator ($\mathcal{I} : x \to -x$) gives the standard orbifold, while the inclusion of the orientation reversal gives us an orientfold. Thus for a manifold M, an example of an orbifold is $M/\mathcal{I}$, while an example of an orientfold is $M/\mathcal{I}\Omega$. Orbifolds and orientfolds give rise to different effects such as non-vanishing vacuum diagrams which must be cancelled via the inclusions of stacked branes (known as non-Abelian gauge cancellation). One of these non-vanishing vacuum diagrams takes the algebraic form of a Klein bottle, which since it is not cancelled it means there is a non-vanishing charge contribution from the diagram. This causes orientfolds to also carry charge. The simplest example of an orbifold is $S^1/\mathbb{Z}_2$, where S^1 is the 1-sphere (unit circle), and $\mathbb{Z}_2$ is the cyclic group of order 2. Here S^1 has the identification of points via the equivalence relation $y \sim y + 2\pi R$ (R is the radius of S^1), while $\mathbb{Z}_2$ has the identifications $y \sim -y$, for $y \in S^1$. Since we can identify all the points of the top semicircle of S^1 with the bottom semicircle of S^1 via the $\mathbb{Z}_2$ identification, we are left with the top semicircle as the geometric picture for the orbifold $S^1/\mathbb{Z}_2$.

It is noted that often times branes are wrapped (which give rise to non-vanishing winding numbers) compact directions. This allows us to recover lower dimensional theories under compactification[†]. Now that we have defined the fields that exist in our theory and the objects which they charge (that both act as energy sources contributing towards the compactification conditions), we can move onto an important symmetry of the theory's action: supersymmetry.

5.4 Supersymmetry

Within the framework of QFT there are a number of subtleties that cause problems with modelling the universe. These conflicts are resolved via imposing the supersymmetry (susy) constraint in the action of the theory. Before delving into the framework of supersymmetry, we briefly outline some problems are that resolved by it.

In renormalizable quantum field theories, such as a φ^4 scalar field theory in 3+1 dimensions, the zero point or vacuum energy does not cancel and diverges:

[†] For example, one could wrap a 2D plane around S^1 to recover a cylinder. When compactifying along the S^1 direction we are left with a line. An example of such in string theory would be recovering the $F1$ string in type IIA theory by compactifying an $M2$ brane occurring in M-theory (which is wrapped along S^1) along the S^1 direction.

$$\langle E\rangle = \int_{-\infty}^{\infty} d^4k\ \sqrt{k^2+m^2} = \int_{0}^{\infty} dk\ dk_0\ k^2\sqrt{|k|^2-k_0^2+m^2}\ \longrightarrow \infty. \tag{52}$$

Here k^2 is the modulus of the 4-momentum k^μ, and m is the mass of the bosonic scalar field φ. The fact that the vacuum energy isn't controlled is problematic as it constrains the value of the cosmological constant Λ, which in turn determines how the universe expands in cosmological theories. We certainly don't want to have a diverging Λ as it would imply a singularity in the rate of expansion which is not the case for our universe. In a supersymmetric theory however, as seen before, the DOFs of the bosonic and fermionic sectors match up. The zero point energy (with $\hbar = 1$) would have contributions from the fermionic and bosonic sector of the theory:

$$\langle E\rangle = \frac{1}{2}\left[\sum_k \omega_k^{(b)} - \sum_k \omega_k^{(f)}\right]. \tag{53}$$

These frequencies $\{\omega_k^{(b,f)}\}$ depend on the field masses $(\omega_k^{(b,f)})^2 = (k^{(b,f)})^2 + m^{(b,f)})$ and since supersymmetry requires all field masses to be the same (and that the bosonic and fermionic sector of the theory is the same), the sum exactly cancels out in the computation of the vacuum energy and gives us a finite value of 0. Another problem in which supersymmetry proved to be useful in is the Coleman-Mandula theorem. This states that an internal symmetry generated by some operator T_α for a symmetry group G can *only* be combined with a Poincaré (extended Lorentz) group $\mathbb{R}^{3,1} \rtimes O(1,3)$ in a trivial way [23]. This means that no commutators or anti-commutators of T_α can produce a generator of the Poincaré group $K_{\alpha\beta}$. The theorem tells us that combining the symmetry groups would have to be a direct product of the two, taking the form $G \times (\mathbb{R}^{3,1} \rtimes O(1,3))$. This condition can be bypassed, however, with the generators of the susy algebra Q_α^L which anticommute to produce a generator of the Poincaré group: $\{Q_\alpha^L, Q_\beta^M\} = \delta^{LM} K_{\alpha\beta}$ [24]. We now we move onto the supersymmetric formalism.

The action of some symmetry group G on some field A is denoted as $\delta_G A$, which tells us how the field transforms under the group. In a supersymmetric theory, the action has a continuous symmetry by which a field can be transformed into a functional combination others, such as $\delta_G A = F[B, C]$, for fields (A, B, C). The generators of the susy algebra are known as *supercharges*, where if the system does not break supersymmetry, there is a maximum of 32 *real* supercharges. Furthermore, the theory of supersymmetry makes use of *supermanifolds* which are other generalizations of manifolds with extra fermionic (Grassmannian-valued) dimensions $(\theta, \bar{\theta})$. The bosonic dimensions of the manifold are the normal spatial direction x^μ [25]. The extra degrees of freedom require us to have *supercoordinates* and *supercovariant* derivatives defined over the superspace, these are given by:

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} = x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\beta}\bar{\theta}^\beta, \quad D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\beta}\bar{\theta}^\beta\partial_\mu. \tag{54}$$

Here x^μ are the usual spacetime coordinates (with bosonic DOFs), θ^α are the spacetime-extension coordinates (with fermionic DOFs), $(\sigma^\mu)_{\alpha\beta}$ are the usual Pauli spin matrices, and the extra term in the supercoordinate is analogous to adding an extra complex dimension to a real space. It is noted the above is for $\mathcal{N} = 1$ susy in 4 dimensions, where $\mathcal{N}$ refers to the number of set of Grassmanian variables in the theory. For higher order susy theories, the supercoordinates and supercovariant derivatives receive additional terms for the new fermionic DOFs. Another piece of convention for susy theories is the conjugate $(\psi^\alpha)^* \equiv \bar{\psi}_{\dot{\alpha}}$, (although we won't be putting the dots on the indices in this paper), which is not to be confused with the usual Dirac conjugate in QFT, i.e. $\bar{\psi} \equiv \psi^\dagger \gamma^0$. To get a better understanding of the spectrum of states or fields allowed in a susy theory (which are grouped up as *supermultiplets*), we will briefly motivate $\mathcal{N} = 1, 2$ susy following the work by Gaumé & Hassan [26].

We begin by looking at the scalar/chiral supermultiplet Φ in $\mathcal{N} = 1$ susy, which is a multiplet of fields that have the associated supersymmetry of the action. The multiplet is defined by the constraint of the form $\bar{D}_\alpha \Phi = 0$. With this constraint in mind we can write down the chiral multiplet as the addition of the following fields:

$$\Phi(y) = \varphi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y), \tag{55}$$

where we have y as our supercoordinate, a scalar field φ, a chiral fermionic field ψ, and an auxiliary field F required for the off-shell closure of the algebra. The fields in this multiplet are *superpartners* of each other, all required to complete the susy algebra. Here we have written the multiplet as an addition of fields instead of grouping them up as a tuple (φ, ψ, F), which will be useful in writing a susy-invariant action integrand. Additionally, we also consider the vector multiplet of the theory satisfying the condition $V = V^\dagger$, and making use of the Wess-Zumino gauge [26], we recover the following multiplet:

$$V(y) = -\theta\sigma^\mu\bar{\theta}A_\mu(y) + i\theta^2\bar{\theta}\bar{\lambda}(y) - i\bar{\theta}^2\theta\lambda(y) + \frac{1}{2}\theta^2\bar{\theta}^2 D(y), \tag{56}$$

where we have a gauge field A_μ, a pair of non-chiral fermionic fields $(\lambda, \bar{\lambda})$ and another auxiliary field D. Now, to find the dynamics of these fields via their equations of motion, we must define an action that is supersymmetrically invariant which we can vary with functional variations δ. To construct this invariant action (as a sum of the chiral and vector multiplet actions) we consider how the susy transformation acts on the different coordinates of the supermanifold, given by:

$$x^\mu \to x^\mu + i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}, \quad \theta \to \theta + \xi, \tag{57}$$

for transformation parameters $(\xi, \bar{\xi})$. Much like how Lorentz transformations are rotations over a Lorentzian space, analogously supersymmetric transformations are rotations over a superspace. Using the above we can construct an invariant action for the chiral multiplet Φ, written as:

$$\begin{aligned} S_\Phi &= \int d^4x\ d^2\theta\ d^2\bar{\theta}\ \Phi^\dagger\Phi + \int d^4x\ \left(d^2\theta\ W[\Phi] + d^2\bar{\theta}\ \bar{W}[\Phi^\dagger]\right) \\ &= \int d^4x\ \left(\partial_\mu\varphi\partial^\mu\varphi - i\bar{\psi}\sigma^\mu\partial_\mu\psi + F^\dagger F\right) + V_s[\Phi]. \end{aligned} \tag{58}$$

Here the first three terms are the usual kinetic terms of the action for the fields, while $V_s[\Phi]$ is the superpotential term describing the interactions of fields in the theory with arbitrary holomorphic functionals of the chiral multiplet $W[\Phi]$. Now, for the vector multiplet we can additionally define the gauge invariant, non-Abelian field strength tensor as:

$$\begin{aligned} W_\alpha &= \frac{1}{8}\bar{D}^2 e^{2V} D_\alpha e^{-2V} \\ &= T^a \left[-i\lambda^a_\alpha + \theta_\alpha D^a - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu)\theta_\alpha F^a_{\mu\nu} + \theta^2\sigma^\mu D_\mu\bar{\lambda}^a\right], \end{aligned} \tag{59}$$

where T^a are the generators of the underlying symmetry group, $F^a_{\mu\nu} = \partial_\mu A^a_\alpha - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$ is the non-Abelian curvature 2-form, $D_\mu = \partial_\mu + f^{abc} A^b_\mu$ is the usual covariant derivative, and f^{abc} structure constants of the continuous symmetry group. With this we can construct a susy invariant action for the vector multiplet which takes the form:

$$\begin{aligned} S_V &= -\frac{1}{4}\int d^4x \ \left[d^2\theta \ \mathrm{tr}(W_\alpha W^\alpha) + d^2\bar{\theta} \ \mathrm{tr}(\bar{W}_\alpha \bar{W}^\alpha)\right] \\ &= \int d^4x \ \left(-\frac{1}{4}\,\mathrm{tr}\, F^{\mu\nu}F_{\mu\nu} + \frac{i}{4} F_{\mu\nu} \star F^{\mu\nu} - \mathrm{tr}\big(i\lambda\sigma^\mu \lambda \nabla_\mu \bar{\lambda}\big) + \frac{1}{2}\,\mathrm{tr}\, D^2\right). \end{aligned} \tag{60}$$

Here we have recovered super Yang-Mills (SYM) $\mathcal{N} = 1$ action, where the first term is the Yang-Mills action, the second term is the instanton/Chern-Simons action, the third term is the Dirac action, and the last term is the auxiliary field action. Note we implicitly sum over the group indices a with the use of the traces. We can conveniently rewrite this action with the complex structure $J = idz^\mu \otimes \partial_\mu - idz^{\bar{\mu}} \otimes \partial_{\bar{\mu}}$ (where $z^{\bar{\mu}}$ is shorthand notation for $\bar{z}^\mu$) of the manifold we integrate over (again explored in the lectures by Candelas [6]). It turns out a convenient choice is the complex torus that has the following complex structure τ:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2_{\mathrm{YM}}}, \tag{61}$$

where θ is an angle prefactor relating to the CP problem in topological Yang-Mills theory, and g_{YM} is the Yang-Mills coupling constant. This can be interpreted as a complex coupling constant of the theory as both (θ, g) are coupling constants that run under the renormalization group. The choice of the complex torus allows us to specify the construction of a gauge theory (by g_{YM}), via the fibration of a torus. This is particularly useful in $\mathcal{N} = 2$ SYM theory, also known as Seiberg-Witten theory. Finally, the way to mix the chiral and vector multiplets is through the addition of the term $\mathrm{tr}\Big(\Phi^\dagger e^{-2gV}\Phi\Big)$. With this we can write the full $\mathcal{N} = 1$ action as the following:

$$S = \int d^4x \ d^2\theta \ d^2\bar{\theta} \ \ \mathrm{tr}\big(\Phi^\dagger e^{-2gV}\Phi\big) + \int d^4x \ \ \mathrm{tr}\ \big(d^2\theta\ W[\Phi] + d^2\bar{\theta}\ \bar{W}[\Phi^\dagger]\big) + \mathrm{Im}\left[\tau \int d^4x \ d^2\theta \ \ \mathrm{tr}(W^\alpha W_\alpha)\right]. \tag{62}$$

Here Im denotes taking the imaginary part of the integral. Similarly, for $\mathcal{N} = 2$ with new fermionic DOFs $(\tilde{\theta}, \bar{\tilde{\theta}})$, we have the same action as above but the extra susy constraints require us to remove the superpotential term. We can also write it in the language of $\mathcal{N} = 2$ by defining the chiral supermultiplet for $\mathcal{N} = 2$ as:

$$\Psi(\tilde{y}) = \Phi(\tilde{y}) + \sqrt{2}\tilde{\theta}^\alpha W_\alpha(\tilde{y}) + \tilde{\theta}^\alpha \tilde{\theta}_\alpha G(\tilde{y}), \tag{63}$$

where G is yet again another auxiliary field, $\tilde{y}^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} + i\tilde{\theta}\sigma^\mu\bar{\tilde{\theta}}$ is the $\mathcal{N} = 2$ supercoordinate, and the other fields are as defined previously. With this we can write the action for $\mathcal{N} = 2$ susy which is exactly equivalent as the action above without the superpotential term:

$$S = \frac{1}{2}\,\mathrm{Im}\left[\tau \int d^4x \ d^2\theta \ d^2\bar{\theta} \ \ \mathrm{tr}\,\Psi^2\right]. \tag{64}$$

From the actions written above we can vary them via $\delta S = 0$ to recover the equations of motions of the fields, and thus their dynamics. With these motivations for supersymmetry in the theory we consider, we can move onto the last and most important prerequisite through which we recover lower dimensional theories: compactification.

5.5 Compactification

The last prerequisite which plays a fundamental role in the framework of superstring theory is *compactification*. This method allows us to generate new theories or generalize current ones by compactifying higher dimensional objects or manifolds. This can be done typically via writing the manifold of the ambient spacetime as a *fibration* or *product space* of two manifolds, and taking the vanishing limit of one of the spaces. The advantage of a fibration or a *fiber bundle* — which is a set of manifolds related through a projective mapping π usually written as $A \rtimes B$ for two manifolds A, B — is that we can define more general/consistent fields & connections as the sections of the bundle. This is commonly used in non-Abelian gauge theories such as Yang-Mills/Chern-Simons theory and theories admitting topological defects. A downside of using a fibration, however, is that the equations of motion of fields defined over the fibration (the total field being written as a tensor product of fields defined on each respective space) pick up extra non-linear terms which completely change the dynamics of the field. These extra terms which are a consequence of the fibration construction aren't present when factoring the ambient space manifold as a product space such as $A \times B$. Thus for our purposes we will consider product spaces only when performing compactifications (also referred to as *dimensional reduction*).

More specifically, compactification is a process in general topology where we take a topological space or manifold, usually one of the extra/internal higher dimensions of the theory, and make it into a compact space [27]. The physics definition extends to taking this compact space to vanish in the limit that the parameter which modulates its size vanishes (such as taking the radius of an n-sphere S^n to vanish: $R \to 0$). For the purposes of our analysis, consider an n-dimensional Lorentzian spacetime whose manifold, Ω_n, can have its temporal and spatial parts factorized as a product space: $\Omega_n = \Sigma_{n-1} \times \mathbb{R}_+$, where Σ_{n-1} corresponds to the $n-1$ spatial dimensions and $\mathbb{R}_+$ corresponds to the positive definite, one dimensional time dimension. Ignoring the temporal piece, we can split up the spatial component as another factorized product space of two manifolds of dimension $n-m-1$ and m, respectively, to give:

$$\Sigma_{n-1} = (\Sigma_{n-m-1} \times \Sigma'_m). \tag{65}$$

We can now look at the dynamics of the $n-m-1$ dimensional theory by compactifying over Σ'_m which is done by taking the limit $\Sigma'_m \xrightarrow{R\to 0} 0$, for some size parameter $R(\Sigma'_m)$. In other words, to get lower dimensional theories from higher dimensional ones, we need to dimensionally reduce or compactify over a *compact* manifold. For example, say we want to recover a 3+1 dimensional theory of the universe we live in from a 9+1 dimensional theory of bosonic superstrings. We can write the overall manifold as the product of a 4-dimensional Lorentzian manifold M_4 and a 6-dimensional (3 complex dimensional) Calabi-Yau 3-fold CY_3[†] and take the limit when the size of $CY_3 \to 0$; i.e.:

$$\text{(10D Theory)}\ \ M_4 \times CY_3 \xrightarrow{CY_3 \to 0} M_4\ \ \text{(4D Theory)}. \tag{66}$$

[†]Certain conditions on this 6-dimensional space force it to be a Calabi-Yau manifold. These conditions include Ricci flatness, an integrable complex structure, and SU(3) holonomy. Again, this is covered in Candelas' lectures [6].

This is of course a simplification of an underlying complicated process but will suffice as motivation for future sections which make use of compactification explicitly. We present some geometrical examples of compactification for intuition.

A trivial example of compactification is compactifying a cylinder (S^1 fibred over some interval I) along the S^1 direction and recovering a line which spans the interval. A more involved example is adding a point at infinity away from a complex plane to give a compact Riemann sphere (which can be thought as the inverse process of stereographic projection over a complex plane). This is a complex projective space P_1, where we have compactified C^1[‡] to the sphere S^2. We can think of this more generally where the set of projective manifolds P_n are the compactified forms of C^n to which a hyperplane has been added at infinity [6]. These compact manifolds are particularly useful when constructing Calabi-Yau manifolds. A more elaborate example of compactification is in the construction of a torus from planes. Consider an elliptic curve given by the exact form $\omega = dx/y$ which can be equivalently written as the curve $y^2 = P_4(x)$, where $P_4(x)$ is a polynomial of degree 4. When taking the square root on both sides, there is an ambiguity in the sign in front of the square root, and thus we get a set of two branch cuts for the positive root and two branch cuts for the negative root. The set of branch cuts for the positive root is defined on a plane different from the plane on which the negative roots are defined on. We can connect the set of branch cuts between the two planes parallel to each other, forming a simply-connected topology between the two. Each plane can then be compactified by bringing points from infinity and the resulting object is topologically equivalent to a torus. This process looks like [28]:

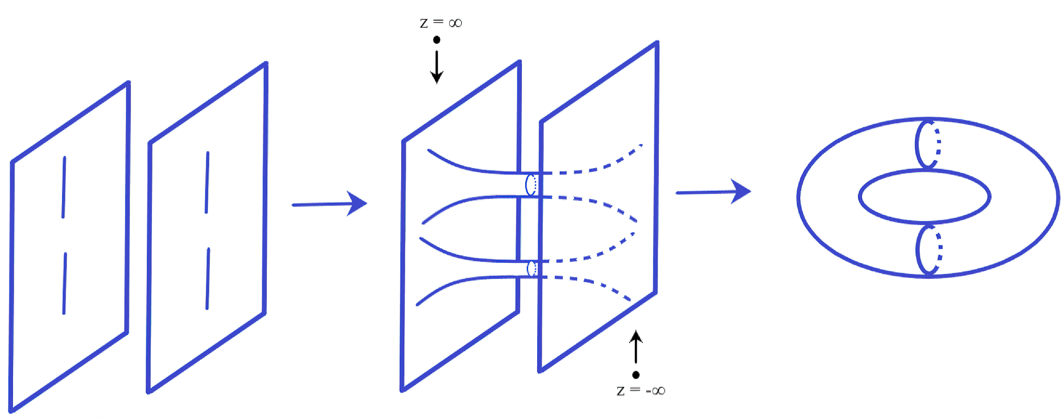

Figure 1: Constructing a torus via compactification. On the left are two parallel 2-dimensional planes with two defined branch cuts over each plane. We connect the branch cuts to get the topology in the middle. Thereafter, we compactify both planes by bringing in points from infinity. The resulting shape is topologically equivalent (diffeomorphic) to a torus $T^2 = S^1 \times S^1$ with genus $g = 1$.

What about recovering known lower dimensional theories from their generalized higher-dimensional counterparts? Consider an 11D string theory with spacetime with coordinates $(x^0, \dots, x^9, x^{11})$ — where we have used a strange convention to take the last dimension as x^{11} and not x^{10} — with a metric field g_{MN}, anti-symmetric axion field C_{MNP}, and a fermionic field ψ_M for indices $(M, N, P) = 0, \dots, 9, 11$. We can consider

‡Some notation has been used here: $C^n \equiv \mathbb{C}^n = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{n \text{ times}}$.

splitting the *spatial* part of our spacetime $\mathbb{R}^{11}$ as a fibration of S^1 and the temporal component $\mathbb{R}_+$:

$$\mathbb{R}^{11} = \mathbb{R}^{10} \times \mathbb{R}_+ = (\mathbb{R}^9 \rtimes S^1) \times \mathbb{R}_+. \tag{67}$$

What possible multiplets of fields can we construct from this after compactifying on S^1? For g_{MN} for example, we can consider all the indices lying on $\mathbb{R}^9$ giving us $g_{\mu\nu}$ where $(\mu, \nu) = 0, \ldots, 9$. We can also consider the case where one index lies in S^1 giving us $g_{\mu,11}$ (which we can identify as a rank 1 tensor gauge field A_μ). Finally, we take both indices to lie on S^1 giving us $g_{11,11}$ (which we can identify with the rank 0 dilaton/scalar field φ). Combining this we get our first multiplet of fields:

$$g_{MN} = g_{\mu\nu} \oplus g_{\mu,11} \oplus g_{11,11} \longrightarrow g_{\mu\nu} \oplus A_\mu \oplus \varphi \tag{68}$$

We can repeat this to similarly get $C_{MNP} = C_{\mu\nu\rho} \oplus C_{\mu\nu,11} \longrightarrow C^{(3)}_{\mu\nu\rho} \oplus B_{\mu\nu}$ for the axion, and $\psi_M = \psi_\mu \oplus \psi_{11} \to \Psi_\mu \oplus \psi$ for the fermion. Notice these are exactly the multiplets for type IIA string theory! Here we took an 11-dimensional theory and compactified along $S^1 \to 0$ to get a known 10-dimensional theory. We may also have compactified over a 2-torus $T^2 = S^1 \times S^1$ to recover type IIB string theory. The more general 11-dimensional theory from which we can derive IIA and IIB is known as M-theory.

Thus we have seen how to recover lower dimensional theories from higher dimensional ones which lines up with the interests of this paper. We want a dS_4 model for our 3+1D universe that arises from compactifying type II string theory. The fact that we use superstring theory allows us to have consistency at different energy scales by having IR/UV mixing, and moreover supersymmetry allows us to avoid the non-cancellation of the vacuum energy of the theory. With this we can focus on the conditions that restrict us from finding de Sitter compactification solutions, which includes the inability to write down an effective Wilsonian action or define a consistent theory vacuum $|\Omega\rangle$. It turns out that these solutions exist as excited states $|\sigma\rangle$ over a flat supersymmetric background (super Minkowski space).

References

[1] J.C. Baez and J.P. Muniain, *Gauge fields, knots, and gravity*, World Scientific, Singapore, 1st ed. (1994).

[2] J. Maldacena and C. Nuñez, *Supergravity description of field theories on curved manifolds and a no go theorem*, International Journal of Modern Physics A **16** (2001) 822.

[3] K. Dasgupta, R. Gwyn, E. McDonough, M. Mia and R. Tatar, *de Sitter Vacua in Type IIB String Theory: Classical Solutions and Quantum Corrections*, Journal of High Energy Physics **2014** (2014) 54.

[4] R. Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. **14** (1965) 57.

[5] H. Bernardo, S. Brahma and M.M. Faruk, *The inheritance of energy conditions: Revisiting no-go theorems in string compactifications*, 2208.09341.

[6] P. Candelas, *Lectures on Complex Manifolds*, Springer-Verlag, 1st ed. (1987).

[7] V. Mukhanov and S. Winitzki, *Introduction to Quantum Effects in Gravity*, Cambridge University Press (2007).

[8] A. Dasgupta, *Quantum Field Theory in Accelerated Frames*, 1508.01940.

[9] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory. Vol. 2: Loop amplitudes, anomalies and phenomenology*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, UK (1987).

[10] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Cambridge University Press, 2nd ed. (2019).

[11] S. Brahma, K. Dasgupta and R. Tatar, *Four-dimensional de sitter space is a glauber-sudarshan state in string theory*, *Journal of High Energy Physics* **2021** (2021) .

[12] S. Brahma, K. Dasgupta, M.-M. Faruk, B. Kulinich, V. Meruliya, B. Pym et al., *Resurgence of a de Sitter Glauber-Sudarshan State: Nodal Diagrams and Borel Resummation*, 2211.09181.

[13] S. Alexander, K. Dasgupta, A. Maji, P. Ramadevi and R. Tatar, *de Sitter State in Heterotic String Theory*, 2303.12843.

[14] F.J. Dyson, *Divergence of perturbation theory in quantum electrodynamics*, *Phys. Rev.* **85** (1952) 631.

[15] D.D. Tong, *University of Cambridge Part III Mathematical Tripos: Lectures on string theory*, *University of Cambridge* (2009) 218.

[16] R. Blumenhagen, D. Lüst and S. Theisen, *Basic concepts of string theory*, Theoretical and mathematical physics, Springer Verlag, Heidelberg (2013).

[17] L. Andrianopoli, R. D'Auria, S. Ferrara, P. Fré, R. Minasian and M. Trigiante, *Solvable lie algebras in type IIA, type IIB and m-theories*, *Nuclear Physics B* **493** (1997) 249.

[18] J.G. Polchinski, *String theory: an introduction to the bosonic string*, Cambridge monographs on mathematical physics, Cambridge University Press, Cambridge ; New York, 1st ed. (1998).

[19] P. Aspinwall, ed., *Dirichlet branes and mirror symmetry*, no. v. 4 in Clay mathematics monographs, American Mathematical Society ; Clay Mathematics Institute, Providence, RI : [Cambridge, MA] (2009).

[20] E. Kiritsis, *String theory in a nutshell*, Princeton University Press, USA, 1st ed. (2019).

[21] A. Adem, J. Leida and Y. Ruan, *Orbifolds and Stringy Topology*, Cambridge University Press (May, 2007), 10.1017/cbo9780511543081.

[22] I. Moerdijk and D.A. Pronk, *Orbifolds, sheaves and groupoids*, *K-Theory* **12** (1997) 3.

[23] S. Coleman and J. Mandula, *All possible symmetries of the s matrix*, *Physical Review* **159** (1967) 1251.

[24] R. Haag, J.T. Łopuszański and M. Sohnius, *All possible generators of supersymmetries of the s-matrix*, *Nuclear Physics B* **88** (1975) 257.

[25] A. Rogers, *Supermanifolds: theory and apllications*, World Scientific, New Jersey (2007).

[26] L. Alvarez-Gaumé and S.F. Hassan, *Introduction to S-duality in $N = 2$ supersymmetric gauge theories (a pedagogical review of the work of seiberg and witten)*, *Fortschritte der Physik/Progress of Physics* **45** (1997) 159.

[27] J.R. Munkres, *Topology*, Pearson, Harlow, 2nd ed. (2014).

[28] D. Greb, B. Sibley, M. Toma and R. Wentworth, *Complex algebraic compactifications of the moduli space of hermitian yang–mills connections on a projective manifold*, *Geometry & Topology* **25** (2021) 1719.